1. Reminder on notations: $\mathbb{R}^n$ stands for the set of all (column) vectors with n entries.

   Recall the definition for the notion of null space:

   > *Let A be an $(m \times n)$-matrix.*
   > *The null space of A is defined to be the set $\{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}$.*
   > *It is denoted by $\mathcal{N}(A)$.*

   In terms of system of linear equations, $\mathcal{N}(A)$ is the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

2. **Definition. (Non-singular matrices and singular matrices.)**

   Let C be a $(p \times p)$-square matrix.

   (a) *C is said to be non-singular if*

   $$\mathcal{N}(C) = \{\mathbf{0}\}.$$

   (b) *C is said to be singular if C is not non-singular.*

Remarks.

(a) We don't talk about non-singularity for non-square matrices.

(b) The statement '$\mathcal{N}(C) = \{\mathbf{0}\}$' in the context of this definition for the notion of *non-singular matrices* is a set equality.

The correct (and formal) way to understand the equality '$\mathcal{N}(C) = \{\mathbf{0}\}$' is that it is a 'short-hand' for this passage:

Both statements (†), (‡) *are true:*

(†) *For any* $\mathbf{x} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \mathcal{N}(C)$ *then* $\mathbf{x} \in \{\mathbf{0}\}$.

(‡) *For any* $\mathbf{x} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \{\mathbf{0}\}$ *then* $\mathbf{x} \in \mathcal{N}(C)$.

Because '$\mathbf{x} \in \{\mathbf{0}\}$' is just a (clumsy) re-formulation of '$\mathbf{x} = \mathbf{0}$', the statement (‡) is trivially true by virtue of the properties of matrix multiplication, and it can be safely ignored.

The essential mathematical content in the statement '$\mathcal{N}(C) = \{\mathbf{0}\}$' is the statement (†).

(c) There are various (direct) re-formulations (according to definition) for the statement '*the* $(p \times p)$*-square matrix* C *is non-singular*'. Dependent on the concrete problem we are dealing with, one of them may be much easier to use than any other.

Remarks.

(a) We don't talk about non-singularity for non-square matrices.

(b) The statement '$\mathcal{N}(C) = \{\mathbf{0}\}$' in the context of this definition for the notion of *non-singular matrices* is a set equality.

The correct (and formal) way to understand the equality '$\mathcal{N}(C) = \{\mathbf{0}\}$' is that it is a 'short-hand' for this passage:

Both statements (†), (‡) *are true:*

(†) *For any* $\mathbf{x} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \mathcal{N}(C)$ *then* $\mathbf{x} \in \{\mathbf{0}\}$.

(‡) *For any* $\mathbf{x} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \{\mathbf{0}\}$ *then* $\mathbf{x} \in \mathcal{N}(C)$.

In the light of this, (†), (‡) are:
(†) For any $x \in \mathbb{R}^p$, if $x \in \mathcal{N}(C)$ then $x = 0$.
(‡) For any $x \in \mathbb{R}^p$, if $x = 0$ then $x \in \mathcal{N}(C)$.

Because '$\mathbf{x} \in \{\mathbf{0}\}$' is just a (clumsy) re-formulation of '$\mathbf{x} = \mathbf{0}$', the statement (‡) is trivially true by virtue of the properties of matrix multiplication, and it can be safely ignored.

The essential mathematical content in the statement '$\mathcal{N}(C) = \{\mathbf{0}\}$' is the statement (†).

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3. **Lemma (1). (Simple re-formulations of the notion of non-singularity.)**

Let C be a $(p \times p)$-square matrix. The statements below are logically equivalent:

($\diamondsuit$) $\mathcal{N}(C) = \{\mathbf{0}\}$.

($\clubsuit$) $\mathbf{0}$ *is the only vector in the null space of* C.

($\heartsuit$) *For any vector* $\mathbf{v} \in \mathbb{R}^p$, *if* $C\mathbf{v} = \mathbf{0}$ *then* $\mathbf{v} = \mathbf{0}$.

($\spadesuit$) *The trivial solution is the only solution for the homogeneous system* $\mathcal{LS}(C, \mathbf{0})$.

Remark. Corresponding to the statement Lemma (1), we may give various (direct) re-formulations (according to definition) for the statement

'*the $(p \times p)$-square matrix C is singular*'.

Dependent on the concrete problem we are dealing with, one of them may be much easier to use than any other:

Let C be a $(p \times p)$-square matrix. The statements below are logically equivalent:

($\sim\diamondsuit$) $\mathcal{N}(C) \neq \{\mathbf{0}\}$.

($\sim\clubsuit$) *There is a non-zero vector in the null space of* C.

($\sim\heartsuit$) *There is some* $\mathbf{v} \in \mathbb{R}^p$ *such that* $\mathbf{v} \neq \mathbf{0}$ *and* $C\mathbf{v} = \mathbf{0}$.

($\sim\spadesuit$) *There is some non-trivial solution for the homogeneous system* $\mathcal{LS}(C, \mathbf{0})$.

3. **Lemma (1). (Simple re-formulations of the notion of non-singularity.)**

Let C be a $(p \times p)$-square matrix. The statements below are logically equivalent:

($\diamondsuit$) $\mathcal{N}(C) = \{\mathbf{0}\}$. ← 'C is non-singular.'

($\clubsuit$) $\mathbf{0}$ *is the only vector in the null space of* C. ← '$\mathcal{N}(C) = \{0\}$' put in plain words.

($\heartsuit$) *For any vector* $\mathbf{v} \in \mathbb{R}^p$, *if* $C\mathbf{v} = \mathbf{0}$ *then* $\mathbf{v} = \mathbf{0}$. ← (†) formulated in a friendly way.

($\spadesuit$) *The trivial solution is the only solution for the homogeneous system* $\mathcal{LS}(C, \mathbf{0})$.
← ($\heartsuit$) stated in words.

Remark. Corresponding to the statement Lemma (1), we may give various (direct) re-formulations (according to definition) for the statement

'*the* $(p \times p)$*-square matrix* C *is singular*'.

Dependent on the concrete problem we are dealing with, one of them may be much easier to use than any other:

Let C be a $(p \times p)$-square matrix. The statements below are logically equivalent:

($\sim\diamondsuit$) $\mathcal{N}(C) \neq \{\mathbf{0}\}$. ← 'C is singular.'

($\sim\clubsuit$) *There is a non-zero vector in the null space of* C. ← '($\clubsuit$) is not true' formulated appropriately.

($\sim\heartsuit$) *There is some* $\mathbf{v} \in \mathbb{R}^p$ *such that* $\mathbf{v} \neq \mathbf{0}$ *and* $C\mathbf{v} = \mathbf{0}$. ← '($\heartsuit$) is not true' formulated appropriately.

($\sim\spadesuit$) *There is some non-trivial solution for the homogeneous system* $\mathcal{LS}(C, \mathbf{0})$.
← '($\spadesuit$) is not true' formulated appropriately.

4. **Examples of non-singular matrices.**

(a) *Let* $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$. *We verify that* A *is non-singular.*

- What to check?
  '$\mathcal{N}(A) = \{\mathbf{0}\}$'.

- What easiest to check?
  'The trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'.

- Detail of argument:

  We find the reduced row-echelon form A' of A by applying row operations:

$$A \longrightarrow \cdots \rightarrow A' = I_2.$$

  (Fill in the detail of the calculations.)
  It follows that the only solution for $\mathcal{LS}(A, \mathbf{0})$ is the trivial solution '$\mathbf{x} = \mathbf{0}$'.
  Hence A is non-singular.

(b) *Let* $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$. *We verify that A is non-singular.*

- What to check?
  '$\mathcal{N}(A) = \{\mathbf{0}\}$'.

- What easiest to check?
  'The trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'.

- How to proceed?

(c) *Let* $A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$. *We verify that A is non-singular.*

- What to check? What easiest to check? How to proceed?

(b) *Let* $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$. *We verify that* A *is non-singular.*

- What to check?
  '$\mathcal{N}(A) = \{\mathbf{0}\}$'.

- What easiest to check?
  'The trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'.

- How to proceed? Obtain a sequence of row operations leading to some reduced row-echelon form A'
  $A = A_1 \to A_2 \to \cdots \to A'$
  See if $A' = I_3$.

(c) *Let* $A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$. *We verify that* A *is non-singular.*

- What to check? What easiest to check? How to proceed?

5. **Examples of singular matrices.**

(a) *Let* $A = \begin{bmatrix} 1 & -5 & 3 \\ 2 & -4 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. *We verify that* A *is singular.*

- What to check?
  '$\mathcal{N}(A) \neq \{\mathbf{0}\}$'.

- What easiest to check?
  'There is a non-trivial solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'.

- Detail of argument:

  We find the reduced row-echelon form A' of A by applying row operations:

$$A \longrightarrow \cdots \to A' = \begin{bmatrix} 1 & 0 & -7/6 \\ 0 & 1 & -5/6 \\ 0 & 0 & 0 \end{bmatrix}$$

  It follows that '$\mathbf{x} = \begin{bmatrix} 7/6 \\ 5/6 \\ 1 \end{bmatrix}$' is a non-trivial solution for $\mathcal{LS}(A, \mathbf{0})$.

  Hence A is singular.

5. **Examples of singular matrices.**

(a) *Let* $A = \begin{bmatrix} 1 & -5 & 3 \\ 2 & -4 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. *We verify that A is singular.*

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- Detail of argument:
  We find the reduced row-echelon form A' of A by applying row operations:

$$A \longrightarrow \cdots \rightarrow A' = \begin{bmatrix} 1 & 0 & -7/6 \\ 0 & 1 & -5/6 \\ 0 & 0 & 0 \end{bmatrix}$$

This says that $\mathcal{LS}(A', 0)$ read:

$$\begin{cases} x_1 \quad - \frac{7}{6}x_3 = 0 \\ x_2 - \frac{5}{6}x_3 = 0 \\ 0 = 0. \end{cases}$$

It follows that '$\mathbf{x} = \begin{bmatrix} 7/6 \\ 5/6 \\ 1 \end{bmatrix}$' is a non-trivial solution for $\mathcal{LS}(A, \mathbf{0})$.

Hence A is singular.

(b) *Let* $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. *We verify that A is singular.*

- What to check?
  '$\mathcal{N}(A) \neq \{\mathbf{0}\}$'.

- What easiest to check?
  'There is a non-trivial solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'.

- How to proceed?

(c) *Let* $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix}$. *We verify that A is singular.*

- What to check? What easiest to check? How to proceed?

(b) *Let* $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. *We verify that A is singular.*

- What to check?
  '$\mathcal{N}(A) \neq \{\mathbf{0}\}$'.

- What easiest to check?
  'There is a non-trivial solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'.

- How to proceed?
  Obtain a sequence of row operations leading to some reduced row-echelon form A'
  $A = A_1 \to A_2 \to \cdots \to A'$.
  See if $A' = \begin{bmatrix} & \\ 0 \; 0 \cdots 0 \end{bmatrix}$ with an entire row of 0's at the bottom.

(c) *Let* $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix}$. *We verify that A is singular.*

- What to check? What easiest to check? How to proceed?

6. **Lemma (2). (Sufficiency criterion for non-singularity in terms of matrix multiplication.)**

Let C be a $(p \times p)$-square matrix.

Suppose there exists some $(p \times p)$-square matrix J such that $JC = I_p$.

Then C is non-singular.

Proof.

Let C be a $(p \times p)$-square matrix.

Suppose there exists some $(p \times p)$-square matrix J such that $JC = I_p$.

[Ask: What to check? 'C is non-singular'.

Which formulation is easiest to use?

$$\text{'}\textit{For any } \mathbf{v} \in \mathbb{R}^p, \textit{ if } C\mathbf{v} = \mathbf{0} \textit{ then } \mathbf{v} = \mathbf{0}.\text{'}$$

Now ask: How to proceed?]

Pick any $\mathbf{v} \in \mathbb{R}^p$. Suppose $C\mathbf{v} = \mathbf{0}$. [Try to deduce: '$\mathbf{v} = \mathbf{0}$.']

By assumption $JC = I_p$. Then $(JC)\mathbf{v} = I_p\mathbf{v} = \mathbf{v}$.

Recall that $C\mathbf{v} = \mathbf{0}$. Then $\mathbf{v} = (JC)\mathbf{v} = J(C\mathbf{v}) = J\mathbf{0} = \mathbf{0}$.

[We have successfully deduced '*For any* $\mathbf{v} \in \mathbb{R}^p$, *if* $C\mathbf{v} = \mathbf{0}$ *then* $\mathbf{v} = \mathbf{0}$'.]

It follows that C is non-singular.

7. **Natural questions to ask, as follow-up to Lemma (2).**

(a) The converse of Lemma (2) reads:

Let C be a $(p \times p)$-square matrix.
Suppose C is non-singular.
Then there exists some $(p \times p)$-square matrix J such that $JC = I_p$.

Question. *Is the converse of Lemma (2) true?*

Answer. It will turn out to be a true statement. (But to see this, a lot of work needs to be done first.)

(b) The statement ($\sharp$) is a generalization of Lemma (2):

($\sharp$) *Let C be a $(p \times q)$-square matrix.*
Suppose there exists some $(q \times p)$-square matrix J such that $JC = I_q$.
Then $\mathcal{N}(C) = \{\mathbf{0}\}$.

Question. *Is the statement ($\sharp$) true?*

Answer. Yes. (How to prove the answer? Exercise.)

8. **More examples of non-singular matrices.**

(a) I_n is non-singular.

(b) Every permutation matrix is non-singular.

An $(n \times n)$-matrix for which there is exactly one 1 in each row and each column, and every other entry is 0 is called a permutation matrix.

Examples:

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$

(c) Every orthogonal matrix is non-singular.

Why? Recall definition: *An $(n \times n)$-square matrix C is orthogonal if $C^tC = CC^t = I_n$.*

Now what does Lemma (1) say?

(d) Every upper uni-triangular matrix is non-singular. (Reason: Lemma (2).) Examples:

- $A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$.
- $B = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}$.
- $C = \begin{bmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & \delta & \epsilon \\ 0 & 0 & 1 & \eta \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

A square matrix for which all diagonal entries are 1 and all entries below the diagonal are 0 is called an upper uni-triangular matrix.

9. **More examples of singular matrices.**

(a) The zero square matrix is singular.

(b) Every strictly upper triangular matrix is singular. Examples:

- $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$.
- $B = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix}$.
- $C = \begin{bmatrix} 0 & \alpha & \beta & \gamma \\ 0 & 0 & \delta & \epsilon \\ 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

A square matrix for which all diagonal entries and all entries below the diagonal are 0 is called a strictly upper triangular matrix.

(c) Every square matrix with an entire column of 0's is singular.

Illustration through (4×4)-square matrices:

- Suppose $A = \begin{bmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$. We claim that A is singular. How to see this?

  Can we name a non-zero vector $\mathbf{v}$ in $\mathbb{R}^4$ for which $A\mathbf{v} = \mathbf{0}$?

  Yes, we take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $A\mathbf{v} = \mathbf{0}$.

- How about $A = \begin{bmatrix} * & 0 & * & * \\ * & 0 & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{bmatrix}$? Or $A = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & 0 & * \\ * & * & 0 & * \end{bmatrix}$? Or $A = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \end{bmatrix}$?

Question. How about an $(n \times n)$-square matrices whose entries in the j-th column are all 0?

(d) Every square matrix with an entire row of 0's is singular.

Illustration through (4×4)-matrices:

- Suppose $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We claim that A is singular. How to see this?

  Apply Gaussian elimination

$$A \longrightarrow \ldots \longrightarrow A'$$

  to obtain the reduced row-echelon form A' which is row-equivalent to A.

  The bottom row of A' is a row of 0's. So $A' = \begin{bmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

  The rank of A' is at most 3. The homogeneous system $\mathcal{LS}(A', \mathbf{0})$ will have a non-trivial solution, say, '$\mathbf{x} = \mathbf{v}$', which will also be a non-trivial solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

  Therefore A is singular.

- How about $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$? Or $A = \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$? Or $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \\ * & * & * & * \end{bmatrix}$?

Question. How about an $(n \times n)$-square matrices whose entries in the i-th row are all 0?

10. **Lemma (3). (Special role of identity matrix amongst reduced row-echelon forms.)**

Let A be an $(n \times n)$-square matrix.

Suppose A is a reduced row-echelon form.

Then A is non-singular if and only if $A = I_n$.

Remark. Lemma (3) tells us that I_n is the only $(n \times n)$-square matrix which is simultaneously a reduced row-echelon form and a non-singular matrix. Every reduced row-echelon form which is not I_n is singular.

Proof. Let A be an $(n \times n)$-square matrix. Suppose A is a reduced row-echelon form.

- Suppose $A = I_n$. Then A is non-singular.
- Suppose A is non-singular.

  Note that there are r pivot columns in A, where r is the rank of A.

  By definition, $r \leq n$.

Then A reads as

$$\begin{bmatrix} 1 & \cdots & 0 & \cdots\cdots & 0 & \cdots \\ 0 & \cdots & 1 & \cdots\cdots & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \\ 0 & \cdots & 0 & \cdots\cdots & 1 & \cdots \\ \hline & \cdots & & \text{all 0's} & & \cdots \\ & \vdots & & \vdots & & \vdots \\ & \cdots & & \text{all 0's} & & \cdots \end{bmatrix}$$

We claim that $r = n$:

Suppose it were true that $r < n$.

Then, because the number of leading ones is strictly smaller than the number of columns, it would happen that some columns of A would fail to be a pivot column.

Furthermore, because there is the same number of rows as of columns, some rows of A would fail to contain a leading one.

Now it would happen that there was at least one row of 0's in A.

Then A would be singular. Contradiction arises.

Therefore $r = n$ is the only possibility.

Then each column of A is a pivot column. Hence $A = I_n$.

Then A reads as

$$\left[\begin{array}{ccccc} 1 & \cdots & 0 & \cdots\cdots & 0 & \cdots \\ 0 & \cdots & 1 & \cdots\cdots & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \\ 0 & \cdots & 0 & \cdots\cdots & 1 & \cdots \\ \hline & \cdots & & \text{all 0's} & & \cdots \\ & \vdots & & \vdots & & \vdots \\ & \cdots & & \text{all 0's} & & \cdots \end{array}\right]$$

The above tells us that there may be zero row, or one row, or two rows, ..., or n rows of all 0's.

We want to show that there is no row of 0's.

We claim that $r = n$:

Suppose it were true that $r < n$.

Then, because the number of leading ones is strictly smaller than the number of columns, it would happen that some columns of A would fail to be a pivot column.

Furthermore, because there is the same number of rows as of columns, some rows of A would fail to contain a leading one.

Now it would happen that there was at least one row of 0's in A.

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Therefore $r = n$ is the only possibility.

Then each column of A is a pivot column. Hence $A = I_n$.