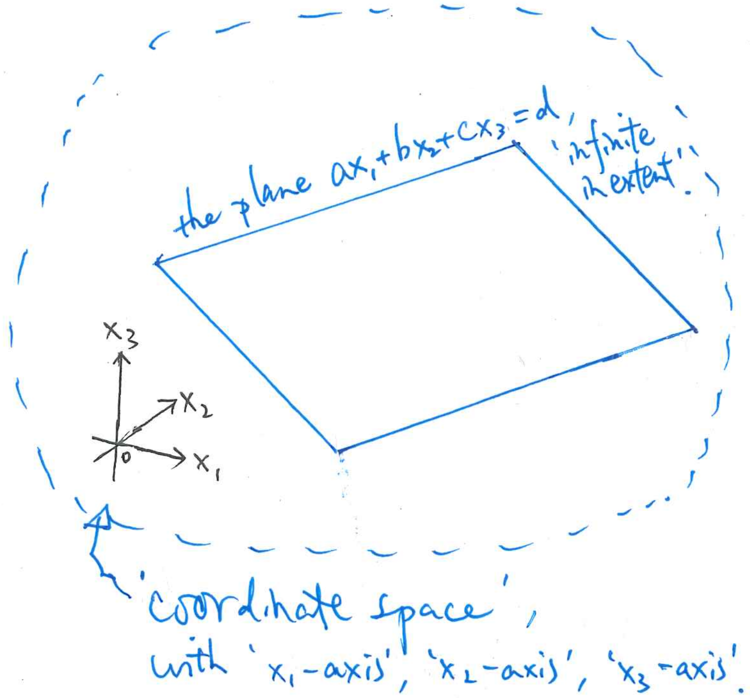
Geometry of a general system of one or two or three linear equations with three unknowns

Suppose a, b, c are real numbers, not all zero.
Suppose d is a real number.
Denote by (U_1) the system

$$ax_1 + bx_2 + cx_3 = d.$$

The solution set P_1 of the system (U_1) corresponds to the 'plane with equation $ax_1 + bx_2 + cx_3 = d$ in the coordinate space'.
The solutions of (U_1) are those and only those points in this plane in the coordinate space.



Now further suppose a', b', c' are real numbers, not all zero.
Suppose d' is a real number.
Denote by (U_2) the system

$$a'x_1 + b'x_2 + c'x_3 = d'$$

and denote by P_2 the solution set of the system (U_2).
Denote by (T_3) the system

$$\begin{cases} ax_1 + bx_2 + cx_3 = d \\ a'x_1 + b'x_2 + c'x_3 = d' \end{cases}$$

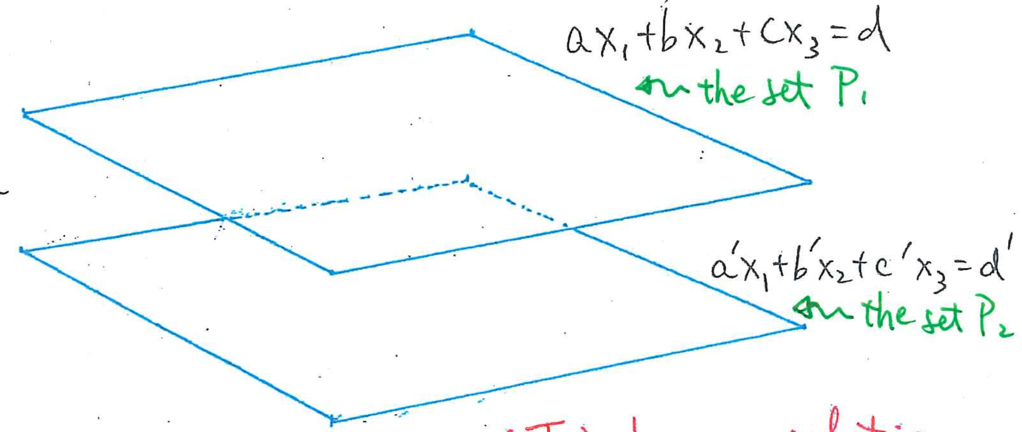
and denote by L_3 the solution set of the system (T_3).

We have $L_3 = P_1 \cap P_2$. There are three mutually exclusive scenarios, dependent on the reduced row-echelon form D' which is row-equivalent to the augmented matrix representation D of the system (T_3).

① Parallel, non-overlapping planes.
$a:b:c = a':b':c'$ and
$a:b:c:d \neq a':b':c':d'$

$D = \begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \end{bmatrix}$

$\to \dots \to D' = \begin{bmatrix} * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

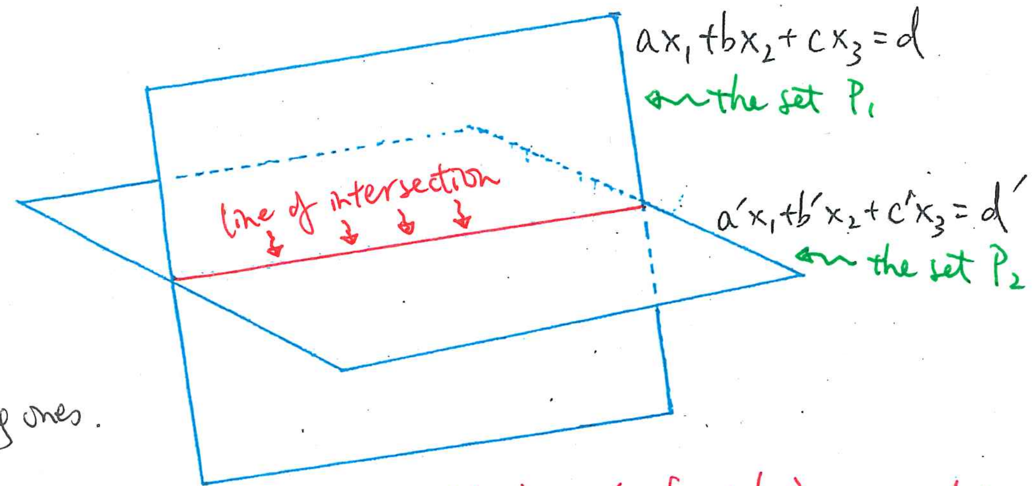


(T_3) has no solution.
$L_3 = P_1 \cap P_2 = \emptyset$.

② Non-parallel planes.
$a:b:c \neq a':b':c'$.

$D = \begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \end{bmatrix}$

$\to \dots \to D' = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \end{bmatrix}$
with two leading ones.

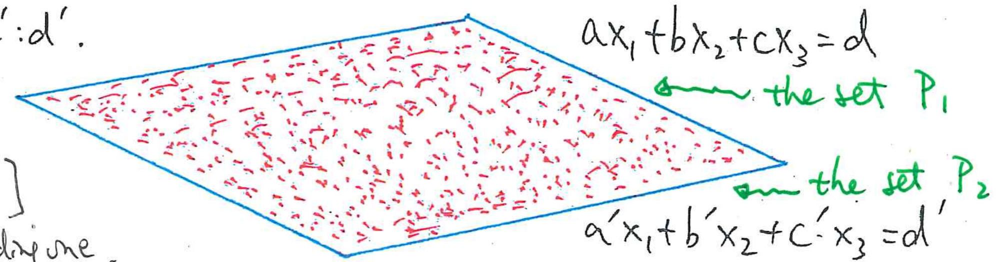


(T_3) has (infinitely) many solutions.
$L_3 = P_1 \cap P_2$ is the set of all points along the line of intersection of the two planes P_1, P_2.

③ Coinciding planes.
$a:b:c:d = a':b':c':d'$.

$D = \begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \end{bmatrix}$

$\to \dots \to D' = \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$
with one leading one.



(T_3) has (infinitely) many solutions.
$L_3 = P_1 \cap P_2 = P_1 = P_2$.

Suppose $a, b, c, d, a', b', c', d', a'', b'', c'', d''$ are real numbers, in which

a, b, c are not all zero,

a', b', c' are not all zero, and

a'', b'', c'' are not all zero.

Denote by (S) the system

$$\begin{cases} a x_1 + b x_2 + c x_3 = d \\ a' x_1 + b' x_2 + c' x_3 = d' \\ a'' x_1 + b'' x_2 + c'' x_3 = d'' \end{cases}$$

and denote by K the solution set of the system (S).

Further introduce the systems below and their respective solution sets:

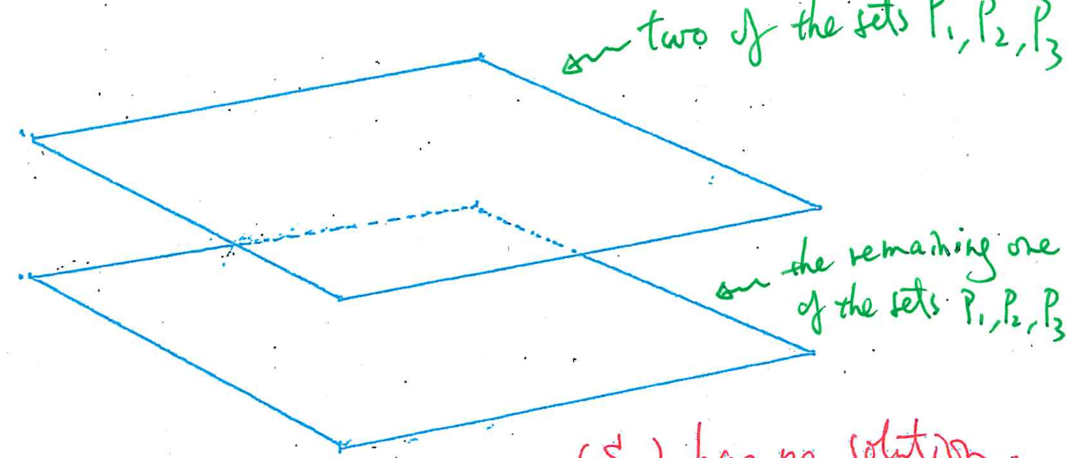
System	Solution set
(U_1) $ax_1 + bx_2 + cx_3 = d$	P_1
(U_2) $a'x_1 + b'x_2 + c'x_3 = d'$	P_2
(U_3) $a''x_1 + b''x_2 + c''x_3 = d''$	P_3
(T_1) $\begin{cases} a'x_1 + b'x_2 + c'x_3 = d' \\ a''x_1 + b''x_2 + c''x_3 = d'' \end{cases}$	L_1
(T_2) $\begin{cases} ax_1 + bx_2 + cx_3 = d \\ a''x_1 + b''x_2 + c''x_3 = d'' \end{cases}$	L_2
(T_3) $\begin{cases} ax_1 + bx_2 + cx_3 = d \\ a'x_1 + b'x_2 + c'x_3 = d' \end{cases}$	L_3

Then $L_1 = P_2 \cap P_3$, $L_2 = P_3 \cap P_1$, $L_3 = P_1 \cap P_2$, and

$K = P_1 \cap P_2 \cap P_3 = L_1 \cap L_2 \cap L_3$.

There are various (mutually exclusive) scenarios, dependent on the reduced row-echelon form D' which is row-equivalent to the augmented matrix representation D of the system (S).

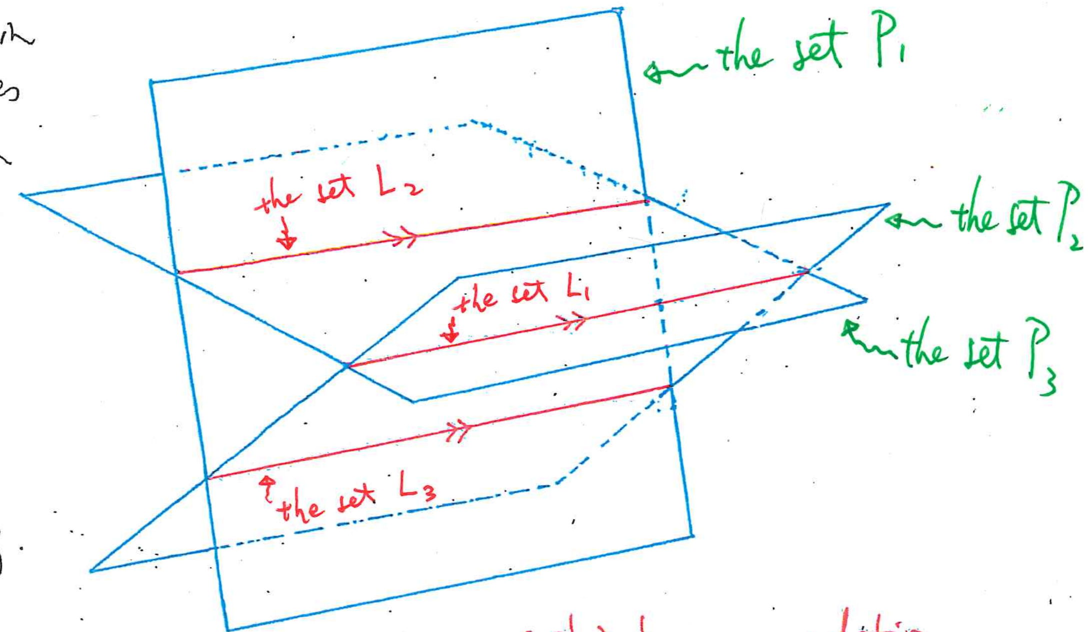
① Amongst the three planes, there are at least two parallel, non-overlapping planes. It does not matter what the remaining plane does.



(S) has no solution.

$K = \phi$.

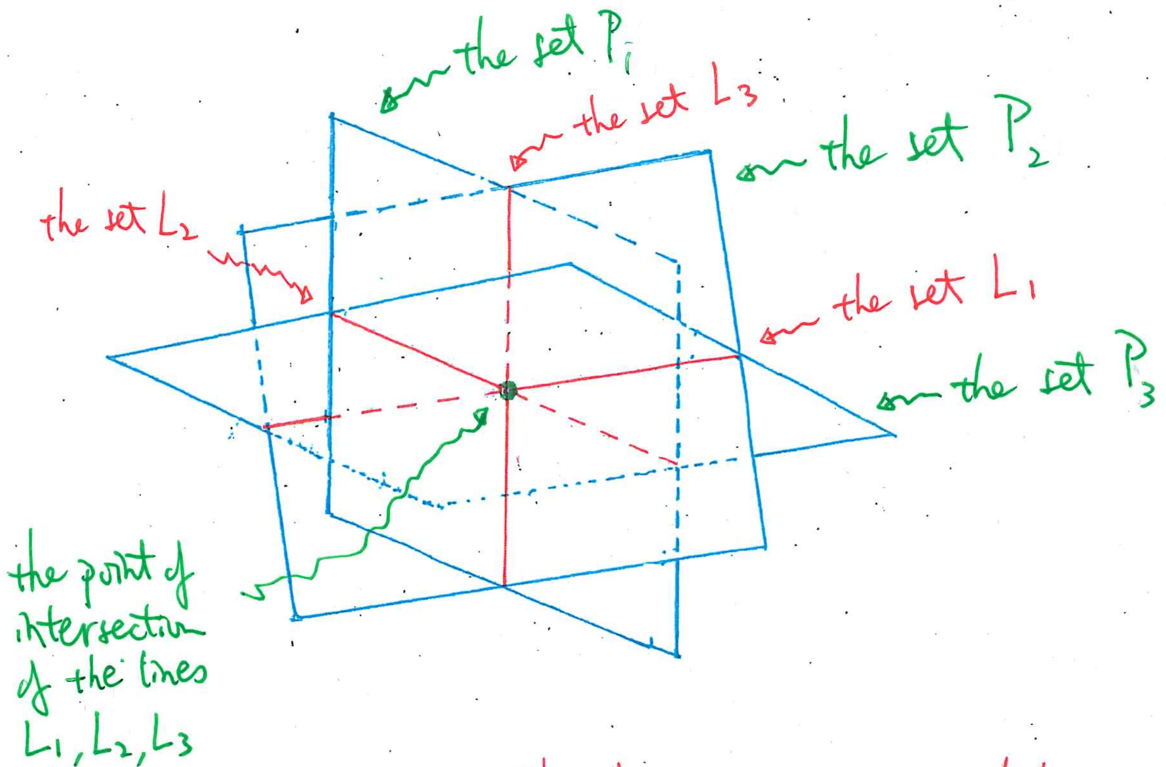
② Every pair in the three planes intersects each other along a line. But the three lines are parallel to each other and are non-overlapping.



(S) has no solution.

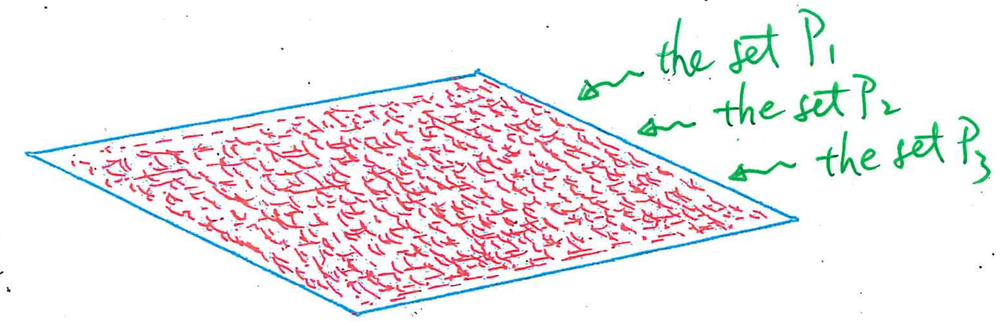
$K = \phi$.

(3) Every pair in the three planes intersects each other along a line. The three lines intersect each other at exactly one point.



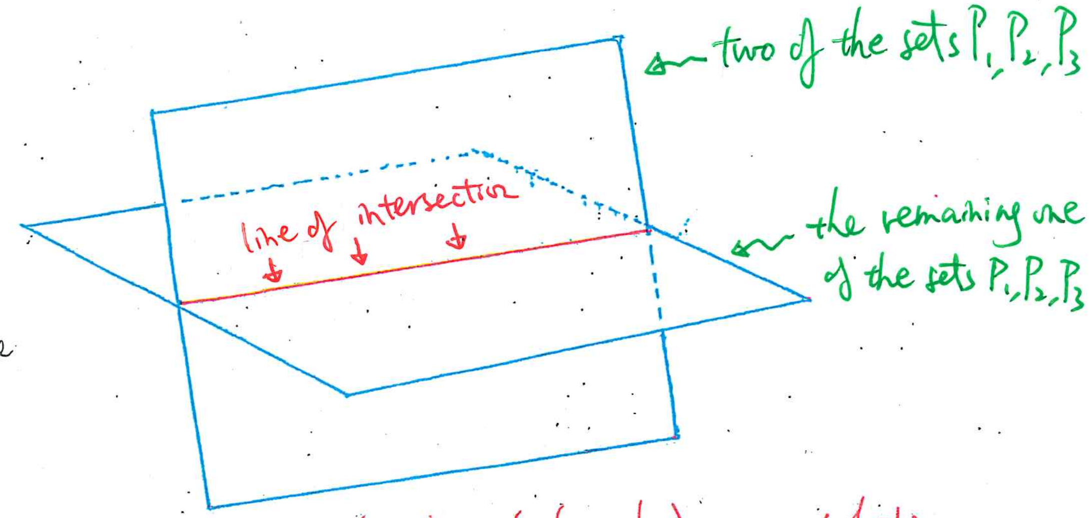
(S) has a unique solution.
K is a set with exactly one element, which is given by the (one and only one) point at which the lines L_1, L_2 and L_3 meet each other.

(5) The three planes coincide with each other.



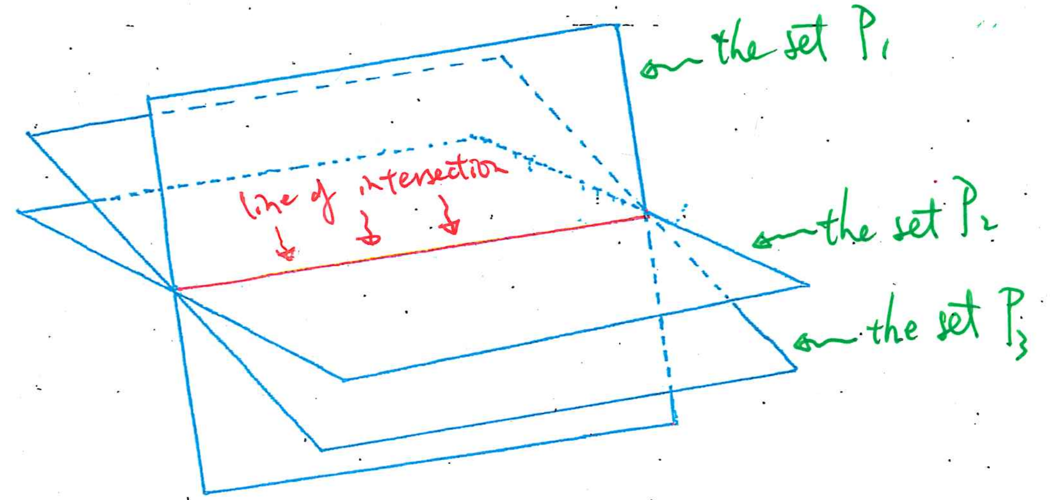
(S) has (infinitely) many solutions.
$K = P_1 \cap P_2 \cap P_3 = P_1 = P_2 = P_3$.
Also, $L_1 = L_2 = L_3 = K$.

(4a) Two of the three planes coincide with each other, and the other plane intersects with it along a line.



(S) has (infinitely) many solutions.
$K = P_1 \cap P_2 \cap P_3 = L_1 = L_2 = L_3$

(4b) The three planes intersect each other along the same line.



(S) has (infinitely) many solutions
$K = P_1 \cap P_2 \cap P_3 = L_1 = L_2 = L_3$