Geometry of a general system of one or two linear equations with two unknowns.

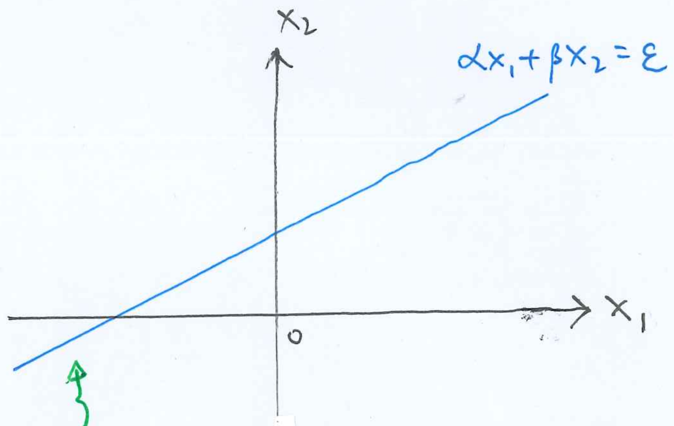
Suppose α, β are real numbers, not both zero.
Suppose ε is a real number.
Denote by (T_1) the system

$$\alpha x_1 + \beta x_2 = \varepsilon .$$

(Matrix presentation for (T_1) is

$$\begin{bmatrix} \alpha & \beta \end{bmatrix} x = \varepsilon .)$$

The solution set L_1 of the system (T_1) corresponds to the 'line with equation $\alpha x_1 + \beta x_2 = \varepsilon$ in the coordinate plane'



The solutions of (T_1) are those and only those points in this line in the coordinate plane.

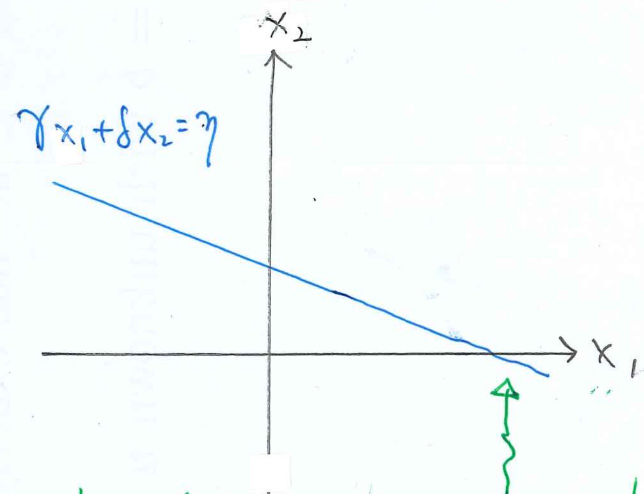
Suppose γ, δ are real numbers, not both zero.
Suppose η is a real number.
Denote by (T_2) the system

$$\gamma x_1 + \delta x_2 = \eta .$$

(Matrix presentation for (T_2) is

$$\begin{bmatrix} \gamma & \delta \end{bmatrix} x = \eta .)$$

The solution set L_2 of the system (T_2) corresponds to the 'line with equation $\gamma x_1 + \delta x_2 = \eta$ in the coordinate plane'.



The solutions of (T_2) are those and only those points in this line in the coordinate plane.

Suppose $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ are real numbers, in which α, β are not both zero, and γ, δ are not both zero.

Denote by (S) the system

$$\begin{cases} \alpha x_1 + \beta x_2 = \varepsilon \\ \gamma x_1 + \delta x_2 = \eta \end{cases}.$$

(Matrix presentation of (S) is

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} x = \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}.$$)

Augmented matrix representation of (S) is

$$C = \begin{bmatrix} \alpha & \beta & \varepsilon \\ \gamma & \delta & \eta \end{bmatrix}.$$

Denote the solution set of (S) by K.

Then $K = L_1 \cap L_2$, in which

L_1 is the solution set of the system (T_1)

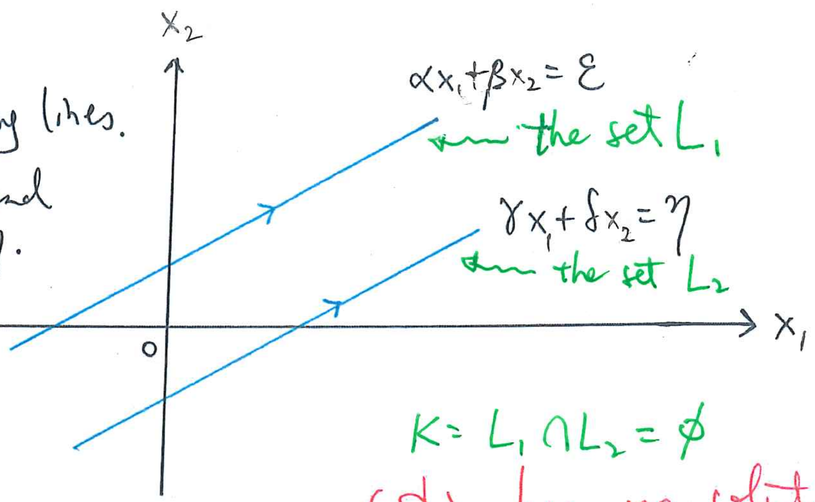
$$\alpha x_1 + \beta x_2 = \varepsilon$$

and L_2 is the solution set of the system (T_2)

$$\gamma x_1 + \delta x_2 = \eta.$$

There are three mutually exclusive scenarios:

① Parallel, non-overlapping lines.
$\alpha:\beta = \gamma:\delta$ and
$\alpha:\beta:\varepsilon \neq \gamma:\delta:\eta$.

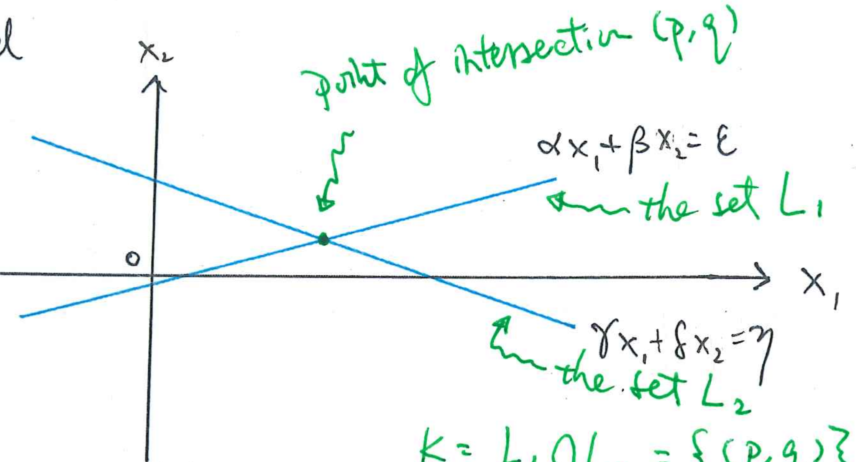


$C \to \cdots\cdots$
$\to C' = \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$
or $C' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$K = L_1 \cap L_2 = \emptyset$

(S) has no solution.

② Non-parallel lines.
$\alpha:\beta \neq \gamma:\delta$.

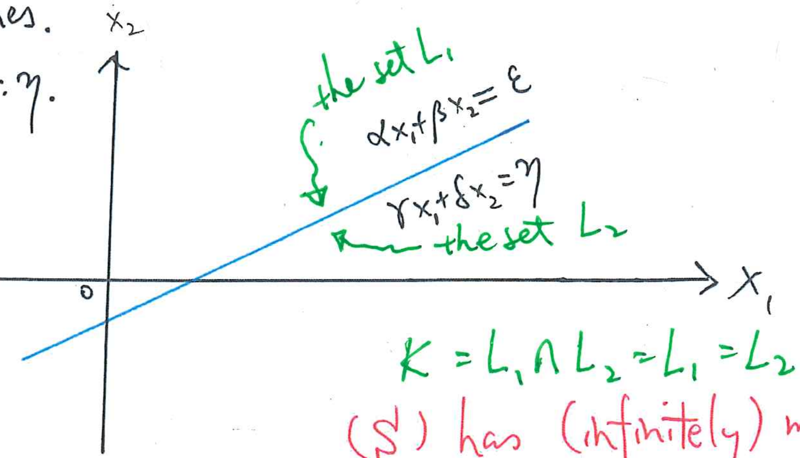


$C \to \cdots\cdots$
$\to C' = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$

$K = L_1 \cap L_2 = \{(p,q)\}$

(S) has a unique solution.

③ Coinciding lines.
$\alpha:\beta:\varepsilon = \gamma:\delta:\eta$.



$C \to \cdots\cdots$
$\to C' = \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$
or $C' = \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$

$K = L_1 \cap L_2 = L_1 = L_2$

(S) has (infinitely) many solutions.