1. **Definition for the notion of 'intersection' (for sets of vectors).**

   Let L, M be sets of vectors in $\mathbb{R}^n$. The intersection of L, M is defined to be the set (of vectors)

   $$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in L \text{ and } \mathbf{x} \in M\}.$$

   It is denoted by $L \cap M$.

   Remarks.

   (a) We are using the method of specification for the construction of $L \cap M$, with 'selection criterion' '$\mathbf{x} \in L$ and $\mathbf{x} \in M$'.

   (b) In plain words, $L \cap M$ is the collection of those, and only those vectors in $\mathbb{R}^n$, which belong to L and belong to M simultaneously.

   (c) This definition can be adapted to general sets of vectors, matrices, or any objects.

   (d) We can extend this definition to that for the intersection of finitely many sets (of vectors):

   Suppose that $K, L, M, N, \cdots$ are sets of vectors in $\mathbb{R}^n$. Then:
   - The intersection $K \cap L \cap M$ is defined to be the set $(K \cap L) \cap M$.
   - The intersection $K \cap L \cap M \cap N$ is defined to be the set $((K \cap L) \cap M) \cap N$.

   Et cetera.

2. **Illustration (1): geometry of the solution sets for a system of two linear equations with two unknowns.**

   (This is a re-packaging of Example (1) in the handout *What is solving a system of linear equations.*)

   Let $A_1 = \begin{bmatrix} 1 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & -1 \end{bmatrix}$, and $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right]$.

   Let $K = \left\{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$, $L_1 = \{\mathbf{x} \in \mathbb{R}^2 : A_1\mathbf{x} = 3\}$, and $L_2 = \{\mathbf{x} \in \mathbb{R}^2 : A_2\mathbf{x} = 4\}$.

   (a) K is the solution set of the system $A\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

   L_1 is the solution set of the system $A_1\mathbf{x} = 3$.

   L_2 is the solution set of the system $A_2\mathbf{x} = 4$.

   (b) Heuristically we expect that $K = L_1 \cap L_2$. Why?

   - K is the solution set of the system $(S) : \left\{\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 3 \\ 2x_1 & - & x_2 & = & 4 \end{array}\right.$. It is explicitly given by $K = \left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$.

     L_1 is the solution set of the system $(T_1) : x_1 + 3x_2 = 3$.

     L_2 is the solution set of the system $(T_2) : 2x_1 - x_2 = 4$.

     Every solution of (S) turns out to be simultaneously a solution of (T_1) and a solution of (T_2).

     Every solution of (T_1) which is simultaneously a solution of (T_2) will turn out to be a solution (S).

   (c) We verify $K = L_1 \cap L_2$ according to definition:

   - Suppose $\mathbf{x} \in K$. [We ask: Is it true that $\mathbf{x} \in L_1 \cap L_2$?]

     Then, by the definition of K and A, A_1, A_2, we have $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = A\mathbf{x} = \left[\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right]\mathbf{x} = \left[\begin{array}{c} A_1\mathbf{x} \\ \hline A_2\mathbf{x} \end{array}\right]$.

     Hence $A_1\mathbf{x} = 3$ and $A_2\mathbf{x} = 4$.

     Since $A_1\mathbf{x} = 3$, we have $\mathbf{x} \in L_1$. Since $A_2\mathbf{x} = 4$, we have $\mathbf{x} \in L_2$.

     Now we have $\mathbf{x} \in L_1$ and $\mathbf{x} \in L_2$ simultaneously. Therefore $\mathbf{x} \in L_1 \cap L_2$.
   - Suppose $\mathbf{x} \in L_1 \cap L_2$. [Ask: Is it true that $\mathbf{x} \in K$?]

     Then, by the definition of intersection, we have $\mathbf{x} \in L_1$ and $\mathbf{x} \in L_2$.

     Now, by the definition of L_1 and A_1, we have $A_1\mathbf{x} = 3$.

     Also, by the definition of L_2 and A_2, we have $A_2\mathbf{x} = 4$.

     Then, by the definition of A, we have $A\mathbf{x} = \left[\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right]\mathbf{x} = \left[\begin{array}{c} A_1\mathbf{x} \\ \hline A_2\mathbf{x} \end{array}\right] = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

     Therefore, by the definition of K, we have $\mathbf{x} \in K$.
   - It follows that $K = L_1 \cap L_2$.

   (d) How to interpret the equality $K = L_1 \cap L_2$ geometrically?

   - First recall that a vector, say, $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$ identified as the point (v_1, v_2) in the coordinate plane.

- L_1 corresponds to the line with equation $x_1 + 3x_2 = 3$ in the coordinate plane, regarded as a set of points on the plane.
  L_2 corresponds to the line with equation $2x_1 - x_2 = 4$ in the coordinate plane, regarded as a set of points on the plane.
  K corresponds to the set $\{(3, -1)\}$. The only point in this set, namely, $(3, -1)$, is where the lines L_1 and L_2 intersect with each other on the coordinate plane.
- In this manner the equality $K = L_1 \cap L_2$ encodes the application of the graphical method for solving the system of linear equations (S).

3. **Geometry of a general system of one or two linear equations with two unknowns.**

   In school maths we learnt from the topic '*graphical method for solving two simultaneous (linear) equations with two unknown* that there would be three scenarios to take care:

   - Scenario one.
     The two lines in the coordinate plane corresponding to the respective equations are distinct and parallel. They do not intersect each other. 'Hence' the system has no solution.
   - Scenario two.
     The two lines in the coordinate plane corresponding to the respective equations have different slope. They intersect each other at one and only one point. 'Hence' the system has exactly one solution, which correspond to the point of intersection for two lines.
   - Scenario three.
     The two lines in the coordinate plane corresponding to the respective equations are the same line. They overlap each other. 'Hence' the system has infinitely many solutions, corresponding to the various points in the overlapping lines.

   Here we re-package these descriptions in what we have learnt about matrices and vectors and in set notations.

   (a) Suppose α, β are real numbers, not both zero. Suppose ϵ is a real number.
   Then the solution set L_1 of the system $(T_1) : \alpha x_1 + \beta x_2 = \epsilon$ (or $\begin{bmatrix} \alpha & \beta \end{bmatrix}\mathbf{x} = \epsilon$) corresponds to the line with equation $\alpha x_1 + \beta x_2 = \epsilon$ on the coordinate plane.

   (b) Also suppose γ, δ are real numbers, not both zero. Suppose η is a real number.
   The solution set L_2 of the system $(T_2) : \gamma x_1 + \delta x_2 = \eta$ (or $\begin{bmatrix} \gamma & \delta \end{bmatrix}\mathbf{x} = \eta$) corresponds to the line with equation $\gamma x_1 + \delta x_2 = \eta$ on the coordinate plane.
   Now consider the system of linear equations $(S) : \left\{ \begin{array}{rcrcl} \alpha x_1 & + & \beta x_2 & = & \epsilon \\ \gamma x_1 & + & \delta x_2 & = & \eta \end{array} \right.$ (or $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\mathbf{x} = \begin{bmatrix} \epsilon \\ \eta \end{bmatrix}$), and its solution set K.
   We can deduce that $K = L_1 \cap L_2$ according to the definition.
   Write $C = \begin{bmatrix} \alpha & \beta & \epsilon \\ \gamma & \delta & \eta \end{bmatrix}$, and denote by C' the reduced row-echelon form which is row-equivalent to C.
   There are three mutually exclusive scenarios; they cover all possibilities:

   i. Suppose $\alpha : \beta = \gamma : \delta$ and $\alpha : \beta : \epsilon \neq \gamma : \delta : \eta$.
   Then $C' = \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $C' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   Therefore (S) is inconsistent: it has no solution.
   In this situation, L_1, L_2 are two distinct parallel lines, and K is the empty set.

   ii. Suppose $\alpha : \beta \neq \gamma : \delta$. Then $C' = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$.
   Therefore (S) is consistent, with a unique solution.
   In this situation, L_1, L_2 are two lines with distinct slopes, and K is the set with exactly one element, which is the point of intersection of L_1 and L_2.

   iii. Suppose $\alpha : \beta : \epsilon = \gamma : \delta : \eta$.
   Then $C' = \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ or $C' = \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$.
   Therefore (S) is consistent, with infinitely many solutions.
   In this situation, L_1, L_2 are two overlapping lines, and K is the same as each of L_1, L_2.

   Remark. Illustration (1) is an example of the description in (b.ii) here.

4. We are going to generalize the description of the geometry of a general system of one or two linear equations with two unknowns to that of a general system of one or two or three linear equations with three unknowns.

   First we extract a useful idea about the manipulations of matrices and vectors from Illustration (1). The idea is formulated in Lemma (1), and Theorem (2).

5. **Lemma (1).**

Let A be an $(m \times n)$-matrix, and A_1, A_2 be matrices with n columns and with m_1, m_2 rows respectively.

Suppose $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right]$.

Let $\mathbf{b}$ *be a vector in* $\mathbb{R}^m$, *and* $\mathbf{b}_1, \mathbf{b}_2$ *be vectors in* $\mathbb{R}^{m_1}$, $\mathbb{R}^{m_2}$ *respectively.*

Suppose $\mathbf{b} = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \end{array}\right]$.

Denote by K the solution set of the system $\mathcal{LS}(A,\ \mathbf{b})$. Denote by L_1 the solution set of the system $\mathcal{LS}(A,\ \mathbf{b}_1)$. Denote by L_2 the solution set of the system $\mathcal{LS}(A,\ \mathbf{b}_2)$.

Then the statements below hold:

(a) *Suppose* $\mathbf{v} \in \mathbb{R}^n$. *Then the equality* $A\mathbf{v} = \left[\begin{array}{c} A_1\mathbf{v} \\ \hline A_2\mathbf{v} \end{array}\right]$ *holds.*

(b) *Suppose $\mathbf{v} \in \mathbb{R}^n$. Then '$\mathbf{x} = \mathbf{v}$' is a solution of system $\mathcal{LS}(A,\ \mathbf{b})$ if and only if for each $h = 1, 2$, '$\mathbf{x} = \mathbf{v}$' is a solution of the system $\mathcal{LS}(A,\ \mathbf{b_h})$.*

(c) $K = L_1 \cap L_2$.

Proof. Extract from Illustration (1) the relevant calculations.

6. **Theorem (2), as a corollary to Lemma (1).**

Let A be an $(m \times n)$-matrix, and $A_1, A_2, \cdots, A_s$ be matrices with n columns and with $m_1, m_2, \cdots, m_s$ rows respectively.

Suppose $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_s \end{array}\right]$.

Let $\mathbf{b}$ *be a vector in* $\mathbb{R}^m$, *and* $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_s$ *be vectors in* $\mathbb{R}^{m_1}$, $\mathbb{R}^{m_2}$, ..., $\mathbb{R}^{m_s}$ *respectively.*

Suppose $\mathbf{b} = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \vdots \\ \hline \mathbf{b}_s \end{array}\right]$.

Denote by K the solution set of the system $\mathcal{LS}(A,\ \mathbf{b})$, and for each h, denote by L_h the solution set of the system $\mathcal{LS}(A,\ \mathbf{b}_h)$.

Then the statements below hold:

(a) *Suppose* $\mathbf{v} \in \mathbb{R}^n$. *Then the equality* $A\mathbf{v} = \left[\begin{array}{c} A_1\mathbf{v} \\ \hline A_2\mathbf{v} \\ \hline \vdots \\ \hline A_s\mathbf{v} \end{array}\right]$ *holds.*

(b) *Suppose $\mathbf{v} \in \mathbb{R}^n$. Then '$\mathbf{x} = \mathbf{v}$' is a solution of system $\mathcal{LS}(A,\ \mathbf{b})$ if and only if for each $h = 1, 2, \cdots, s$, '$\mathbf{x} = \mathbf{v}$' is a solution of the system $\mathcal{LS}(A,\ \mathbf{b_h})$.*

(c) $K = L_1 \cap L_2 \cap \cdots \cap L_h$.

Proof. Apply mathematical induction, with the help of Lemma (1).

7. **Illustration (2): geometry of the solution set for a system of three linear equations with three unknowns.**

(This is a re-packaging of Example (2) in the handout *What is solving a system of linear equations.*)

Let $A_1 = [\ 1 \quad 2 \quad 2\ ]$, $A_2 = [\ 1 \quad 3 \quad 3\ ]$, $A_3 = [\ 2 \quad 6 \quad 5\ ]$, and $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline A_3 \end{array}\right]$.

Let $K = \left\{ \mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$, $P_1 = \{\mathbf{x} \in \mathbb{R}^3 : A_1\mathbf{x} = 4\}$, $P_2 = \{\mathbf{x} \in \mathbb{R}^3 : A_2\mathbf{x} = 5\}$, $P_3 = \{\mathbf{x} \in \mathbb{R}^3 : A_3\mathbf{x} = 6\}$.

(a) K is the solution set of the system $A\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

P_1 is the solution set of the system $A_1\mathbf{x} = 4$.
P_2 is the solution set of the system $A_2\mathbf{x} = 5$.
P_3 is the solution set of the system $A_3\mathbf{x} = 6$.

(b) Heuristically we expect that $K = P_1 \cap P_2 \cap P_3$. Why?

- K is the solution set of the system $(S) : \left\{ \begin{array}{ccccccc} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array} \right.$. It is explicitly given by $K = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$.
  P_1 is the solution set of the system $(U_1) : x_1 + 2x_2 + 2x_3 = 4$.
  P_2 is the solution set of the system $(U_2) : x_1 + 3x_2 + 3x_3 = 5$.
  P_3 is the solution set of the system $(U_3) : 2x_1 + 6x_2 + 5x_3 = 6$.
  Every solution of (S) turns out to be simultaneously a solution of (U_1) and a solution of (U_2) and a solution of (U_3).
  Every solution of (U_1) which is simultaneously a solution of (U_2) as well as a solution of (U_3) will turn out to be a solution (S).

(c) We verify $K = P_1 \cap P_2 \cap P_3$ according to definition:

- Suppose $\mathbf{x} \in K$. [We ask: Is it true that $\mathbf{x} \in P_1 \cap P_2 \cap P_3$?]
  Then, by the definition of K and A, A_1, A_2, A_3, we have $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = A\mathbf{x} = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline A_3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} A_1\mathbf{x} \\ \hline A_2\mathbf{x} \\ \hline A_3\mathbf{x} \end{bmatrix}$.
  Hence $A_1\mathbf{x} = 4$ and $A_2\mathbf{x} = 5$ and $A_3\mathbf{x} = 6$.
  Since $A_1\mathbf{x} = 4$, we have $\mathbf{x} \in P_1$. Since $A_2\mathbf{x} = 5$, we have $\mathbf{x} \in P_2$. Since $A_3\mathbf{x} = 6$, we have $\mathbf{x} \in P_3$.
  Now we have $\mathbf{x} \in P_1$ and $\mathbf{x} \in P_2$ and $\mathbf{x} \in P_3$ simultaneously.
  Since $\mathbf{x} \in P_1$ and $\mathbf{x} \in P_2$, we have $\mathbf{x} \in P_1 \cap P_2$. Furthermore, since $\mathbf{x} \in P_3$, we have $\mathbf{x} \in P_1 \cap P_2 \cap P_3$.
- Suppose $\mathbf{x} \in P_1 \cap P_2 \cap P_3$. [Ask: Is it true that $\mathbf{x} \in K$?]
  Then, by the definition of intersection, we have $\mathbf{x} \in P_1 \cap P_2$ and $\mathbf{x} \in P_3$. Since $\mathbf{x} \in P_1 \cap P_2$, we have $\mathbf{x} \in P_1$ and $\mathbf{x} \in P_2$.
  Now, by the definition of P_1 and A_1, we have $A_1\mathbf{x} = 4$.
  By the definition of P_2 and A_2, we have $A_2\mathbf{x} = 5$.
  By the definition of P_3 and A_3, we have $A_3\mathbf{x} = 6$.
  Then, by the definition of A, we have $A\mathbf{x} = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline A_3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} A_1\mathbf{x} \\ \hline A_2\mathbf{x} \\ \hline A_3\mathbf{x} \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.
  Therefore, by the definition of K, we have $\mathbf{x} \in K$.
- It follows that $K = P_1 \cap P_2 \cap P_3$.

(d) How to interpret the equality $K = P_1 \cap P_2 \cap P_3$ geometrically?

- First recall that a vector, say, $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$ identified as the point (v_1, v_2, v_3) in the coordinate space.
- P_1 corresponds to the plane with equation $x_1 + 2x_2 + 2x_3 = 4$ in the coordinate space, regarded as a set of points in the space.
  P_2 corresponds to the plane with equation $x_1 + 3x_2 + 3x_3 = 5$ in the coordinate space, regarded as a set of points in the space.
  P_3 corresponds to the plane with equation $2x_1 + 6x_2 + 5x_3 = 6$ in the coordinate space, regarded as a set of points in the space.
- P_1 and P_2 intersects each other along some line L_3 in the coordinate space.
  L_3 is in fact the solution set of the equation $\begin{bmatrix} A_1 \\ \hline A_2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.
  It is explicitly given by $L_3 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \middle| \ r \in \mathbb{R} \right\}$.
  P_2 and P_3 intersects each other along some line L_1 in the coordinate space.
  L_1 is in fact the solution set of the equation $\begin{bmatrix} A_2 \\ \hline A_3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.
  It is explicitly given by $L_1 = \left\{ \begin{bmatrix} -7 \\ 0 \\ 4 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \middle| \ s \in \mathbb{R} \right\}$.
  P_3 and P_1 intersects each other along some line L_2 in the coordinate space.
  L_2 is in fact the solution set of the equation $\begin{bmatrix} A_1 \\ \hline A_3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.
  It is explicitly given by $L_2 = \left\{ \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$.

- L_1, L_2, L_3 have different 'directions'. However, they meet each other at exactly one point in the coordinate space. That point is $(2, -3, 4)$, which is the one and only one element of the set K.

8. **Geometry of a general system of one or two or three linear equations with three unknowns.**

   (a) Suppose a, b, c are real numbers, not all zero. Suppose d is a real number.

   Then the solution set P_1 of the system $(U_1) : ax_1 + bx_2 + cx_3 = d$ (or $\begin{bmatrix} a & b & c \end{bmatrix}\mathbf{x} = d$) corresponds to the plane with equation $ax_1 + bx_2 + cx_3 = d$ in the coordinate space.

   (b) Also suppose a', b', c' are real numbers, not all zero. Suppose d' is a real number.

   The solution set P_2 of the system $(U_2) : a'x_1 + b'x_2 + c'x_3 = d'$ (or $\begin{bmatrix} a' & b' & c' \end{bmatrix}\mathbf{x} = d'$) corresponds to the plane with equation $a'x_1 + b'x_2 + c'x_3 = d'$ in the coordinate space.

   Now consider the system of linear equations $(T_3) : \left\{ \begin{array}{ccccccc} ax_1 & + & bx_2 & + & cx_3 & = & d \\ a'x_1 & + & b'x_2 & + & c'x_3 & = & d' \end{array} \right.$ (or $\begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix}\mathbf{x} = \begin{bmatrix} d \\ d' \end{bmatrix}$), and its solution set L_3.

   We can deduce that $L_3 = P_1 \cap P_2$ according to the definition.

   Write $D = \begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \end{bmatrix}$, and denote by D' the reduced row-echelon form which is row-equivalent to D.

   There are three mutually exclusive scenarios; they cover all possibilities:

   i. Suppose $a : b : c = a' : b' : c'$ and $a : b : c : d \neq a' : b' : c' : d'$.
   Then $D' = \begin{bmatrix} * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, in which the first row contains a leading one, amongst the $*$'s.
   Therefore (T_3) is inconsistent: it has no solution.
   In this situation, P_1, P_2 are two distinct parallel planes, and L_3 is the empty set.

   ii. Suppose $a : b : c \neq a' : b' : c'$.
   Then $D' = \begin{bmatrix} * & * & * & \star \\ 0 & * & * & \star \end{bmatrix}$, in which each row contains a leading one, amongst the $*$'s.
   Therefore (T_3) is consistent, with infinitely many solutions.
   In this situation, P_1, P_2 are two distinct planes with intersecting along a common line which is the line L_3.

   iii. Suppose $a : b : c : d = a' : b' : c' : d'$. .
   Then $D' = \begin{bmatrix} * & * & * & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$, in which the first row contains a leading one, amongst the $*$'s.
   Therefore (T_3) is consistent, with infinitely many solutions.
   In this situation, P_1, P_2 are two overlapping lines, and L_3 is the same as each of P_1, P_2.

   (c) Further suppose a'', b'', c'' are real numbers, not all zero. Suppose d'' is a real number.

   The solution set P_3 of the system $(U_3) : a''x_1 + b''x_2 + c''x_3 = d''$ (or $\begin{bmatrix} a'' & b'' & c'' \end{bmatrix}\mathbf{x} = d''$) corresponds to the plane with equation $a''x_1 + b''x_2 + c''x_3 = d''$ in the coordinate space.

   Consider the system of linear equations $(T_1) : \left\{ \begin{array}{ccccccc} a'x_1 & + & b'x_2 & + & c'x_3 & = & d' \\ a''x_1 & + & b''x_2 & + & c''x_3 & = & d'' \end{array} \right.$ (or $\begin{bmatrix} a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix}\mathbf{x} = \begin{bmatrix} d' \\ d'' \end{bmatrix}$), and its solution set L_1.

   Also consider the system of linear equations $(T_2) : \left\{ \begin{array}{ccccccc} ax_1 & + & bx_2 & + & cx_3 & = & d \\ a''x_1 & + & b''x_2 & + & c''x_3 & = & d'' \end{array} \right.$ (or $\begin{bmatrix} a & b & c \\ a'' & b'' & c'' \end{bmatrix}\mathbf{x} = \begin{bmatrix} d \\ d'' \end{bmatrix}$), and its solution set L_2.

   We can deduce that $L_1 = P_2 \cap P_3$, $L_2 = P_1 \cap P_3$ according to the definition.

   (d) Consider the system of linear equations $(S) : \left\{ \begin{array}{ccccccc} ax_1 & + & bx_2 & + & cx_3 & = & d \\ a'x_1 & + & b'x_2 & + & c'x_3 & = & d' \\ a''x_1 & + & b''x_2 & + & c''x_3 & = & d'' \end{array} \right.$ (or $\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix}\mathbf{x} = \begin{bmatrix} d \\ d' \\ d'' \end{bmatrix}$), and its solution set K.

   We can deduce that $K = L_1 \cap L_2 \cap L_3 = P_1 \cap P_2 \cap P_3$ according to the definition.

   Write $C = \begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{bmatrix}$, and denote by C' the reduced row-echelon form which is row-equivalent to C.

   There are various mutually exclusive scenarios; they cover all possibilities:

i. Suppose $C' = \begin{bmatrix} * & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and the first row contains a leading one, amongst the $*$'s.

Then (S) is inconsistent. Therefore (S) has no solution.
In this situation, two of P_1, P_2, P_3 are parallel planes, at least two of which are distinct from each other. K is the empty set.

ii. Suppose $C' = \begin{bmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and each of the first two rows contains a leading one, amongst the $*$'s.

In this situation, one possibility is that two of P_1, P_2, P_3 are distinct parallel planes, and the third intersect respectively each of those two planes along a line.
Another possibility is that P_1, P_2 intersect along the line L_3, and P_2, P_3 intersect along the line L_1, and P_1, P_3 intersect along the line L_2, and L_1, L_2, L_3 are three distinct parallel lines in the space.
In any case K is the empty set.

iii. Suppose $C' = \begin{bmatrix} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{bmatrix}$,

Then (S) is consistent, and has a unique solution.
In this situation, P_1, P_2 intersect along the line L_3, and P_2, P_3 intersect along the line L_1, and P_1, P_3 intersect along the line L_2, and L_1, L_2, L_3 are three distinct lines meeting at a common point in space. This point is the one and only one element of K.

iv. Suppose $C' = \begin{bmatrix} * & * & * & \star \\ 0 & * & * & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$, in which each of the first two rows of C' contains a leading one, amongst the $*$'s.

Then (S) is consistent, and has infinitely many solutions.
In this situation, P_1, P_2 intersect along the line L_3, and P_2, P_3 intersect along the line L_1, and P_1, P_3 intersect along the line L_2, and L_1, L_2, L_3 are the same line in space. The set K is also this line in space.

v. Suppose $C' = \begin{bmatrix} * & * & * & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, in which the first row of D' contains a leading one, amongst the $*$'s.

Then (S) is consistent, and has infinitely many solutions.
In this situation, P_1, P_2, P_3 are the same plane in space. The set K is also this plane in space.

Remark. Illustration (2) is an example of the description in (d.iii) here. Illustration (3) below is an example for one possibility in the description in (d.ii). Illustration (4) below is an example of the description in (d.iv).

9. **Illustration (3).**

(This is a re-packaging of Example (3) in the handout *What is solving a system of linear equations.*)

Consider the system (S) : $\left\{ \begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 7 \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 \end{array} \right.$.

Write $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ 1 & 3 & -5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$. So the solution set of (S) is the set $K = \{\mathbf{x} \in R^3 : A\mathbf{x} = \mathbf{b}\}$.

Write $A_1 = [\ 1 \quad -1 \quad 1\ ]$, $A_2 = [\ 3 \quad -2 \quad 1\ ]$, $A_3 = [\ 1 \quad 3 \quad -5\ ]$, and

$$
\begin{aligned}
P_1 &= \{\mathbf{x} \in \mathbb{R}^3 : A_1\mathbf{x} = 2\}, \\
P_2 &= \{\mathbf{x} \in \mathbb{R}^3 : A_2\mathbf{x} = 7\}, \\
P_3 &= \{\mathbf{x} \in \mathbb{R}^3 : A_3\mathbf{x} = 3\}, \\
L_1 &= \left\{\mathbf{x} \in \mathbb{R}^3 : \left[\frac{A_2}{A_3}\right]\mathbf{x} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}\right\}, \\
L_2 &= \left\{\mathbf{x} \in \mathbb{R}^3 : \left[\frac{A_1}{A_3}\right]\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}, \\
L_3 &= \left\{\mathbf{x} \in \mathbb{R}^3 : \left[\frac{A_1}{A_2}\right]\mathbf{x} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}\right\}
\end{aligned}
$$

So $L_1 = P_2 \cap P_3$, $L_2 = P_1 \cap P_3$, $L_3 = P_2 \cap P_3$, and $K = P_1 \cap P_2 \cap P_3 = L_1 \cap L_2 \cap L_3$.

Note that P_1, P_2, P_3 are three planes in the coordinate space $\mathbb{R}^3$.

After some manipulation, we find that:

$$\begin{aligned} L_1 &= \left\{ \begin{bmatrix} 27/7 \\ 16/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\} \\ L_2 &= \left\{ \begin{bmatrix} 9/2 \\ 5/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\} \\ L_3 &= \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\} \end{aligned}$$

L_1, L_2, L_3 are distinct parallel lines in space. They do not meet each other.

So K is the empty set.

Indeed, the reduced row-echelon form C' which is row equivalent to $[\; A \mid \mathbf{b} \;]$ is given by

$$C' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So it is expected that $K = \emptyset$.

10. **Illustration (4).**

(This is a re-packaging of Example (4) in the handout *What is solving a system of linear equations.*)

Consider the system (S) : $\left\{ \begin{array}{rcrcrcr} & & x_2 & - & 2x_3 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 \end{array} \right.$.

Write $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ 11 \end{bmatrix}$. So the solution set of (S) is the set $K = \{\mathbf{x} \in R^3 : A\mathbf{x} = \mathbf{b}\}$.

Write $A_1 = [\; 0 \;\; 1 \;\; -2 \;]$, $A_2 = [\; -1 \;\; -2 \;\; 3 \;]$, $A_3 = [\; 2 \;\; 7 \;\; -12 \;]$, and

$$\begin{aligned} P_1 &= \{\mathbf{x} \in \mathbb{R}^3 : A_1\mathbf{x} = 1\}, \\ P_2 &= \{\mathbf{x} \in \mathbb{R}^3 : A_2\mathbf{x} = -4\}, \\ P_3 &= \{\mathbf{x} \in \mathbb{R}^3 : A_3\mathbf{x} = 11\}, \\ L_1 &= \left\{\mathbf{x} \in \mathbb{R}^3 : \left[\frac{A_2}{A_3}\right] \mathbf{x} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}\right\}, \\ L_2 &= \left\{\mathbf{x} \in \mathbb{R}^3 : \left[\frac{A_1}{A_3}\right] \mathbf{x} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}\right\}, \\ L_3 &= \left\{\mathbf{x} \in \mathbb{R}^3 : \left[\frac{A_1}{A_2}\right] \mathbf{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}\right\} \end{aligned}$$

So $L_1 = P_2 \cap P_3$, $L_2 = P_1 \cap P_3$, $L_3 = P_2 \cap P_3$, and $K = P_1 \cap P_2 \cap P_3 = L_1 \cap L_2 \cap L_3$.

Note that P_1, P_2, P_3 are three distinct planes in the coordinate space $\mathbb{R}^3$.

After some manipulation, we find that:

$$L_1 = L_2 = L_3 \quad = \quad \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}$$

L_1, L_2, L_3 the same line in space.

So $K = L$ also.

Indeed, the reduced row-echelon form C' which is row equivalent to $[\; A \mid \mathbf{b} \;]$ is given by

$$C' = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So it is expected that $K = L_1 = L_2 = L_3$.