

0. Reminder on notations: \mathbb{R}^n stands for the set of all (column) vectors with n entries.

1. Definition.

Let A be an $(m \times n)$ -matrix with real entries.

- (a) The system of linear equations $\mathcal{LS}(A, \mathbf{0})$ is called the homogeneous system with coefficient matrix A .
- (b) For each $\mathbf{b} \in \mathbb{R}^m$, the system $\mathcal{LS}(A, \mathbf{0})$ is referred to as the homogeneous system associated to the system of linear equations $\mathcal{LS}(A, \mathbf{b})$.

2. Theorem (1).

Let A be an $(m \times n)$ -matrix. Let $\mathbf{b} \in \mathbb{R}^m$.

- (a) For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, if

$\mathbf{x} = \mathbf{u}$, $\mathbf{x} = \mathbf{v}$ are solutions of $\mathcal{LS}(A, \mathbf{b})$,

then

$\mathbf{x} = \mathbf{v} - \mathbf{u}$ is a solution of $\mathcal{LS}(A, \mathbf{0})$.

- (b) For any $\mathbf{u}, \mathbf{h} \in \mathbb{R}^n$, if

$\mathbf{x} = \mathbf{u}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ and $\mathbf{x} = \mathbf{h}$ is a solution of $\mathcal{LS}(A, \mathbf{0})$,

then

$\mathbf{x} = \mathbf{u} + \mathbf{h}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$.

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The system $Ax = b$
with unknown x in \mathbb{R}^n .

2. Theorem (1).

Let A be an $(m \times n)$ -matrix. Let $\mathbf{b} \in \mathbb{R}^m$.

- (a) For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, if

' $\mathbf{x} = \mathbf{u}$ ', ' $\mathbf{x} = \mathbf{v}$ ' are solutions of $\mathcal{LS}(A, \mathbf{b})$,

then

' $\mathbf{x} = \mathbf{v} - \mathbf{u}$ ' is a solution of $\mathcal{LS}(A, \mathbf{0})$.

The difference between
any two solutions of
 $\mathcal{LS}(A, \mathbf{b})$ is a solution
of $\mathcal{LS}(A, \mathbf{0})$.

- (b) For any $\mathbf{u}, \mathbf{h} \in \mathbb{R}^n$, if

' $\mathbf{x} = \mathbf{u}$ ' is a solution of $\mathcal{LS}(A, \mathbf{b})$ and ' $\mathbf{x} = \mathbf{h}$ ' is a solution of $\mathcal{LS}(A, \mathbf{0})$,

then

' $\mathbf{x} = \mathbf{u} + \mathbf{h}$ ' is a solution of $\mathcal{LS}(A, \mathbf{b})$.

The sum of any solution of
 $\mathcal{LS}(A, \mathbf{b})$ and any solution of
 $\mathcal{LS}(A, \mathbf{0})$ is a solution
of $\mathcal{LS}(A, \mathbf{b})$.

3. Proof of Theorem (1).

Let A be an $(m \times n)$ -matrix. Let $\mathbf{b} \in \mathbb{R}^m$.

(a) Pick any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Suppose

‘ $\mathbf{x} = \mathbf{u}$ ’, ‘ $\mathbf{x} = \mathbf{v}$ ’ are solutions of $\mathcal{LS}(A, \mathbf{b})$.

Then $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$.

Therefore

$$A(\mathbf{v} - \mathbf{u}) = A\mathbf{v} - A\mathbf{u} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Hence ‘ $\mathbf{x} = \mathbf{v} - \mathbf{u}$ ’ is a solution of $\mathcal{LS}(A, \mathbf{0})$.

(b) Pick any $\mathbf{u}, \mathbf{h} \in \mathbb{R}^n$.

Suppose

‘ $\mathbf{x} = \mathbf{u}$ ’ is a solution of $\mathcal{LS}(A, \mathbf{b})$

and

‘ $\mathbf{x} = \mathbf{h}$ ’ is a solution of $\mathcal{LS}(A, \mathbf{0})$.

Then $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{h} = \mathbf{0}$.

Therefore

$$A(\mathbf{u} + \mathbf{h}) = A\mathbf{u} + A\mathbf{h} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Hence ‘ $\mathbf{x} = \mathbf{u} + \mathbf{h}$ ’ is a solution of $\mathcal{LS}(A, \mathbf{b})$.

4. **Theorem (2).** (Set-theoretic re-formulation of Theorem (1).)

Let A be an $(m \times n)$ -matrix.

For each $\mathbf{c} \in \mathbb{R}^m$, denote the solution set of $\mathcal{LS}(A, \mathbf{c})$ by $\Pi_{\mathbf{c}}$. (Note that $\Pi_{\mathbf{0}} = \mathcal{N}(A)$.)

For each $\mathbf{b} \in \mathbb{R}^m$, for each $\mathbf{u} \in \Pi_{\mathbf{b}}$, the equality

$$\Pi_{\mathbf{b}} = \left\{ \mathbf{t} \in \mathbb{R}^n : \begin{array}{l} \text{There exists some } \mathbf{h} \in \mathcal{N}(A) \\ \text{such that } \mathbf{t} = \mathbf{u} + \mathbf{h} \end{array} \right\}$$

holds.

Remark.

What the set equality in the conclusion says is:

(†) For any $\mathbf{v} \in \mathbb{R}^n$, if

\mathbf{v} belongs to the solution set of $\mathcal{LS}(A, \mathbf{b})$

then

there exists some $\mathbf{h} \in \mathcal{N}(A)$ such that $\mathbf{v} = \mathbf{u} + \mathbf{h}$.

(‡) For any $\mathbf{v} \in \mathbb{R}^n$, if

there exists some $\mathbf{h} \in \mathcal{N}(A)$ such that $\mathbf{v} = \mathbf{u} + \mathbf{h}$

then

\mathbf{v} belongs to the solution set of $\mathcal{LS}(A, \mathbf{b})$.

4. Theorem (2). (Set-theoretic re-formulation of Theorem (1).)

Let A be an $(m \times n)$ -matrix.

For each $\mathbf{c} \in \mathbb{R}^m$, denote the solution set of $\mathcal{LS}(A, \mathbf{c})$ by $\Pi_{\mathbf{c}}$. (Note that $\Pi_{\mathbf{0}} = \mathcal{N}(A)$.)

For each $\mathbf{b} \in \mathbb{R}^m$, for each $\mathbf{u} \in \Pi_{\mathbf{b}}$, the equality

$$\Pi_{\mathbf{b}} = \left\{ \mathbf{t} \in \mathbb{R}^n : \begin{array}{l} \text{There exists some } \mathbf{h} \in \mathcal{N}(A) \\ \text{such that } \mathbf{t} = \mathbf{u} + \mathbf{h} \end{array} \right\}$$

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(‡) For any $\mathbf{v} \in \mathbb{R}^n$, if

there exists some $\mathbf{h} \in \mathcal{N}(A)$ such that $\mathbf{v} = \mathbf{u} + \mathbf{h}$

then

\mathbf{v} belongs to the solution set of $\mathcal{LS}(A, \mathbf{b})$.

← This is the collection of those and only those vectors in \mathbb{R}^n which are resultant from adding \mathbf{u} to the various vectors in $\mathcal{N}(A)$. This makes sense without any reference to $\mathcal{LS}(A, \mathbf{b})$.

Further remark.

Some people like to ‘abuse notation’ to present the conclusion in this result as:

For each $\mathbf{b} \in \mathbb{R}^m$, for each $\mathbf{u} \in \Pi_{\mathbf{b}}$, the ‘equality’ $\Pi_{\mathbf{b}} = \mathbf{u} + \Pi_{\mathbf{0}}$ holds.

The ‘equality’ $\Pi_{\mathbf{b}} = \mathbf{u} + \Pi_{\mathbf{0}}$, when put in plain words, is to be interpreted as:

the ‘totality of all solutions of $\mathcal{LS}(A, \mathbf{b})$ ’ can be regarded as

$$\left(\begin{array}{l} \text{one ‘particular solution’} \\ \text{of } \mathcal{LS}(A, \mathbf{b}) \end{array} \right) + \left(\begin{array}{l} \text{totality of all solutions} \\ \text{of } \mathcal{LS}(A, \mathbf{0}) \end{array} \right)$$

5. Illustration (1) of the idea in Theorem (1) and Theorem (2), from school maths.

Let $A = \begin{bmatrix} -1 & 2 \end{bmatrix}$.

For each real number b , $\mathcal{LS}(A, b)$ is simply the one linear equation two unknowns $-x_1 + 2x_2 = b$.

For each b , the solution set of $\mathcal{LS}(A, b)$ is the ‘entire line’ in the coordinate plane.

The null space of A is the line ℓ_0 passing through the origin with slope $\frac{1}{2}$.

Now suppose $b \neq 0$ (for the sake of visual illustration).

(a) The solution set of $\mathcal{LS}(A, b)$ is the line ℓ_b parallel to ℓ_0 and distinct from ℓ_0 , passing through the point $(-b, 0)$.

‘ $(x_1, x_2) = (-b, 0)$ ’ is a solution of $\mathcal{LS}(A, b)$.

(b) Now identify column vectors of size 2 with the points on the coordinate plane in the natural way. We see that:

- \mathbf{v} belongs to ℓ_b if and only if $\mathbf{v} = \begin{bmatrix} -b \\ 0 \end{bmatrix} + \mathbf{h}$ for some $\mathbf{h} \in \mathcal{N}(A)$.

We can visualize this relation geometrically as follows:—

- We can obtain ℓ_b from ℓ_0 by applying to every point of ℓ_0 a ‘translation’ by $\begin{bmatrix} -b \\ 0 \end{bmatrix}$.

(c) In general, given any point (u_1, u_2) on ℓ_b , it happens that:

- \mathbf{v} belongs to ℓ_b if and only if $\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \mathbf{g}$ for some $\mathbf{g} \in \mathcal{N}(A)$.

We can visualize this relation geometrically as follows:—

- We can obtain ℓ_b from ℓ_0 by applying to every point of ℓ_0 a ‘translation’ by $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

(d) For instance, note that ℓ_b passes through $(0, b/2)$.

Then ‘ $(x_1, x_2) = (0, b/2)$ ’ is also a solution of $\mathcal{LS}(A, b)$. It so happens that:

- \mathbf{v} belongs to ℓ_b if and only if $\mathbf{v} = \begin{bmatrix} 0 \\ b/2 \end{bmatrix} + \mathbf{f}$ for some $\mathbf{f} \in \mathcal{N}(A)$.

6. Illustration (2) of the idea in Theorem (1) and Theorem (2).

Let $A = \begin{bmatrix} 1/3 & 1/2 & -1 \end{bmatrix}$.

For each real number b , $\mathcal{LS}(A, b)$ is simply the one linear equation with unknowns $\frac{1}{3}x_1 + \frac{1}{2}x_2 - x_3 = b$.

For each b , the solution set of $\mathcal{LS}(A, b)$ is the ‘entire line’ in the coordinate plane.

The null space of A is the plane Π_0 passing through the origin with ‘normal direction’ parallel to the vector $\begin{bmatrix} 1/3 \\ 1/2 \\ -1 \end{bmatrix}$.

It is explicitly given by $\mathcal{N}(A) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = s \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \text{ for some } s, t \in \mathbb{R} \right\}$.

Now suppose $b \neq 0$ (for the sake of visual illustration).

(a) The solution set of $\mathcal{LS}(A, b)$ is the line ℓ_b parallel to Π_0 and distinct from Π_0 , passing through the point $(3b, 0, 0)$.

‘ $(x_1, x_2, x_3) = (3b, 0, 0)$ ’ is a solution of $\mathcal{LS}(A, b)$.

(b) Now identify column vectors of size 3 with the points on the coordinate space in the natural way. We see that:

- \mathbf{v} belongs to Π_b if and only if $\mathbf{v} = \begin{bmatrix} 3b \\ 0 \\ 0 \end{bmatrix} + \mathbf{h}$ for some $\mathbf{h} \in \mathcal{N}(A)$.

We can visualize this relation geometrically as follows:—

- We can obtain Π_b from Π_0 by applying to every point of Π_0 a ‘translation’ by $\begin{bmatrix} 3b \\ 0 \\ 0 \end{bmatrix}$.

(c) In general, given any point (u_1, u_2, u_3) on Π_b , it happens that:

- \mathbf{v} belongs to Π_b if and only if $\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \mathbf{g}$ for some $\mathbf{g} \in \mathcal{N}(A)$.

We can visualize this relation geometrically as follows:—

- We can obtain Π_b from Π_0 by applying to every point of Π_0 a ‘translation’ by $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

(d) For instance, note that Π_b passes through $(0, 0, -b)$. Then ‘ $(x_1, x_2, x_3) = (0, 0, -b)$ ’ is also a solution of $\mathcal{LS}(A, b)$. It so happens that:

- \mathbf{v} belongs to Π_b if and only if $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix} + \mathbf{f}$ for some $\mathbf{f} \in \mathcal{N}(A)$.

7. **Illustration (3) of the idea in Theorem (1) and Theorem (2).**

$$\text{Let } A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}.$$

For each vector \mathbf{b} in \mathbb{R}^2 , $\mathcal{LS}(A, \mathbf{b})$ is the system of two linear equations with three unknowns

$$\begin{cases} x_1 & - 2x_3 = b_1 \\ & x_2 - 3x_3 = b_2 \end{cases},$$

in which $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

For each b_1, b_2 , the solution set of $\mathcal{LS}(A, b)$ is a ‘line’ in the coordinate space.

The null space of A is the plane $\Lambda_{\mathbf{0}}$ passing through the origin in the ‘direction’ of the vector $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

It is explicitly given by $\mathcal{N}(A) = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

Now suppose $b \neq 0$ (for the sake of visual illustration).

(a) The solution set of $\mathcal{LS}(A, \mathbf{b})$ is the line $\Lambda_{\mathbf{b}}$ parallel to $\Lambda_{\mathbf{0}}$ and distinct from $\Lambda_{\mathbf{0}}$, passing through the point $(b_1, b_2, 0)$.

‘ $(x_1, x_2, x_3) = (b_1, b_2, 0)$ ’ is a solution of $\mathcal{LS}(A, \mathbf{b})$.

(b) Now identify column vectors of size 3 with the points on the coordinate space in the natural way. We see that:

- \mathbf{v} belongs to $\Lambda_{\mathbf{b}}$ if and only if $\mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} + \mathbf{h}$ for some $\mathbf{h} \in \mathcal{N}(A)$.

- We can obtain $\Lambda_{\mathbf{b}}$ from $\Lambda_{\mathbf{0}}$ by applying to every point of $\Lambda_{\mathbf{0}}$ a ‘translation’ by $\begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$.

(c) In general, given any point (u_1, u_2, u_3) on $\Lambda_{\mathbf{b}}$, it happens that:

- \mathbf{v} belongs to $\Lambda_{\mathbf{b}}$ if and only if $\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \mathbf{g}$ for some $\mathbf{g} \in \mathcal{N}(A)$.

We can visualize this relation geometrically as follows:—

- We can obtain $\Lambda_{\mathbf{b}}$ from $\Lambda_{\mathbf{0}}$ by applying to every point of $\Lambda_{\mathbf{0}}$ a ‘translation’ by $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

8. Proof of Theorem (2).

Let $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{u} \in \Pi_{\mathbf{b}}$. By assumption, $A\mathbf{u} = \mathbf{b}$.

Write

$$\Sigma_{\mathbf{b},\mathbf{u}} = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} = \mathbf{u} + \mathbf{h} \text{ for some } \mathbf{h} \in \mathcal{N}(A)\}.$$

[We want to prove $\Pi_{\mathbf{b}} = \Sigma_{\mathbf{b},\mathbf{u}}$.]

- [We want to prove that ‘every vector in $\Pi_{\mathbf{b}}$ belongs to $\Sigma_{\mathbf{b},\mathbf{u}}$.’]

What do we mean to prove, really? ‘For any vector \mathbf{w} , if $\mathbf{w} \in \Pi_{\mathbf{b}}$, then $\mathbf{w} \in \Sigma_{\mathbf{b},\mathbf{u}}$.’]

Pick any vector \mathbf{w} . Suppose $\mathbf{w} \in \Pi_{\mathbf{b}}$.

[Ask: Is it true that $\mathbf{w} \in \Sigma_{\mathbf{b},\mathbf{u}}$?]

By definition, $A\mathbf{w} = \mathbf{b}$.

(Recall that $A\mathbf{u} = \mathbf{b}$ also.)

Take $\mathbf{h} = \mathbf{w} - \mathbf{u}$. Then, by definition, $A\mathbf{h} = \mathbf{0}$. Therefore $\mathbf{h} \in \mathcal{N}(A)$.

Also by definition, $\mathbf{w} = \mathbf{u} + \mathbf{h}$.

Hence $\mathbf{w} \in \Sigma_{\mathbf{b},\mathbf{u}}$.

- [We want to prove that ‘every vector in $\Sigma_{\mathbf{b},\mathbf{u}}$ belongs to $\Pi_{\mathbf{b}}$.’]

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Proof of Theorem (2).

Let $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{u} \in \Pi_{\mathbf{b}}$. By assumption, $A\mathbf{u} = \mathbf{b}$.

Write

$$\Sigma_{\mathbf{b},\mathbf{u}} = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} = \mathbf{u} + \mathbf{h} \text{ for some } \mathbf{h} \in \mathcal{N}(A)\}.$$

[We want to prove $\Pi_{\mathbf{b}} = \Sigma_{\mathbf{b},\mathbf{u}}$.]

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- [We want to prove that ‘every vector in $\Sigma_{\mathbf{b},\mathbf{u}}$ belongs to $\Pi_{\mathbf{b}}$.’]

What do we mean to prove, really? ‘For any vector \mathbf{w} , if $\mathbf{w} \in \Sigma_{\mathbf{b},\mathbf{u}}$, then $\mathbf{w} \in \Pi_{\mathbf{b}}$.’]

Pick any vector \mathbf{w} . Suppose $\mathbf{w} \in \Sigma_{\mathbf{b},\mathbf{u}}$.

[Ask: Is it true that $\mathbf{w} \in \Pi_{\mathbf{b}}$?]

By definition, $\mathbf{w} = \mathbf{u} + \mathbf{h}$ for some $\mathbf{h} \in \mathcal{N}(A)$.

By definition, $A\mathbf{h} = \mathbf{0}$.

(Recall that $A\mathbf{u} = \mathbf{b}$.)

Then $A\mathbf{w} = A(\mathbf{u} + \mathbf{h}) = A\mathbf{u} + A\mathbf{h} = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

Therefore $\mathbf{w} \in \Pi_{\mathbf{b}}$.

It follows that the equality $\Pi_{\mathbf{b}} = \Sigma_{\mathbf{b},\mathbf{u}}$ holds.

9. Illustration (4) of the idea in Theorem (1) and Theorem (2).

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}.$$

Consider the system $\mathcal{LS}(A, \mathbf{b})$ and its associated homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

(a) The respective augmented matrix representations of $\mathcal{LS}(A, \mathbf{b})$ and $\mathcal{LS}(A, \mathbf{0})$ are $C_{\mathbf{b}} = [A | \mathbf{b}]$ and $C_{\mathbf{0}} = [A | \mathbf{0}]$.

The reduced row-echelon forms $C'_{\mathbf{b}}, C'_{\mathbf{0}}$ which are row-equivalent to $C_{\mathbf{b}}, C_{\mathbf{0}}$ respectively are given by

$$C'_{\mathbf{b}} = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right], \quad C'_{\mathbf{0}} = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{Write } \mathbf{h}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{h}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

(b) The solution set of $\mathcal{LS}(A, \mathbf{b})$ is given by $\Pi_{\mathbf{b}} = \{\mathbf{u} + c_1\mathbf{h}_1 + c_2\mathbf{h}_2 | c_1, c_2 \in \mathbb{R}\}$.

Note that ' $\mathbf{x} = \mathbf{u}$ ' is a (particular) solution of $\mathcal{LS}(A, \mathbf{b})$.

(c) The solution set of $\mathcal{LS}(A, \mathbf{0})$ (or in other words, the null space of A) is given by $\mathcal{N}(A) = \{c_1\mathbf{h}_1 + c_2\mathbf{h}_2 | c_1, c_2 \in \mathbb{R}\}$.

(d) The difference of any vector in $\Pi_{\mathbf{b}}$ from any vector in $\Pi_{\mathbf{b}}$ is the vector $\alpha_1\mathbf{h}_1 + \alpha_2\mathbf{h}_2$ for some numbers α_1, α_2 . It belongs to $\mathcal{N}(A)$.

(e) The sum of any vector in $\Pi_{\mathbf{b}}$ and any vector in $\mathcal{N}(A)$ is the vector $\beta_1\mathbf{h}_1 + \beta_2\mathbf{h}_2$ for some numbers β_1, β_2 . It belongs to $\Pi_{\mathbf{b}}$.

(f) The relation between $\Pi_{\mathbf{b}}$ and $\mathcal{N}(A)$ can be visualized geometrically as follows:—

- We can obtain $\Pi_{\mathbf{b}}$ from $\mathcal{N}(A)$ by applying to every point of $\mathcal{N}(A)$ a 'translation' by \mathbf{u} .

10. Theorem (3).

Suppose A is an $(m \times n)$ -matrix. Then:

(1) $\mathcal{LS}(A, \mathbf{0})$ is consistent, with ' $\mathbf{x} = \mathbf{0}$ ' as a solution for the system.

(Or equivalently, $\mathcal{N}(A)$ contains some vector in \mathbb{R}^n , namely $\mathbf{0}$.)

(2) The statements below are logically equivalent:

(2a) $\mathbf{0}$ is the only vector in $\mathcal{N}(A)$.

(2b) $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

(2c) For each vector $\mathbf{b} \in \mathbb{R}^m$, if $\mathcal{LS}(A, \mathbf{b})$ is consistent then $\mathcal{LS}(A, \mathbf{b})$ has a unique solution.

(3) The statements below are logically equivalent:

(3a) $\mathcal{N}(A)$ contains some vector in \mathbb{R}^n other than $\mathbf{0}$.

(3b) $\mathcal{LS}(A, \mathbf{0})$ has at least two solutions.

(3c) For each vector $\mathbf{b} \in \mathbb{R}^m$, if $\mathcal{LS}(A, \mathbf{b})$ is consistent then $\mathcal{LS}(A, \mathbf{b})$ has at least two solutions.

Proof. Exercise. (Apply Theorem (1).)