0. Reminder on notations: $\mathbb{R}^n$ stands for the set of all (column) vectors with n entries.

1. **Definition. (Null space of a matrix.)**

   Let A be an $(m \times n)$-matrix.

   (a) *The system of linear equations $\mathcal{LS}(A, \mathbf{0})$ is called the homogeneous system with coefficient matrix A.*

   (b) *The solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ is called the null space of A. It is denoted by $\mathcal{N}(A)$.*

   Remark.

   First of all, recall that

   '$\mathbf{x} = \mathbf{u}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$ if and only if $A\mathbf{u} = \mathbf{0}$.

   Using as 'selection criterion' the equality '$A\mathbf{x} = \mathbf{0}$', we may present the null space of A as a set constructed using the method of specification:

   - Those vectors in $\mathbb{R}^n$ which, upon substitution into the '$\mathbf{x}$' in this 'selection criterion' result in an equality, will be 'collected'.
   - Those vectors in $\mathbb{R}^n$ which, upon substitution into the '$\mathbf{x}$' in this 'selection' do not result in an equality, will be 'discarded'.

   Hence the null space of A is the set $\{\mathbf{u} \in \mathbb{R}^n : A\mathbf{u} = \mathbf{0}\}$.

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1. **Definition. (Null space of a matrix.)**

   The system $Ax = 0$ with unknown x in $\mathbb{R}^n$.

   Let A be an $(m \times n)$-matrix.

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   Hence the null space of A is the set $\{\mathbf{u} \in \mathbb{R}^n : A\mathbf{u} = \mathbf{0}\}$.

Further remark.

How to use the various versions of the definitions?

Always remember, whenever $\mathbf{v} \in \mathbb{R}^n$, the statements below mean the same thing:

(a) $\mathbf{v} \in \mathcal{N}(A)$.

(b) $A\mathbf{v} = \mathbf{0}$.

(c) '$\mathbf{x} = \mathbf{v}$' *is a solution of the homogeneous system* $\mathcal{LS}(A, \mathbf{0})$ *with unknown* $\mathbf{x}$.

To determine $\mathcal{N}(A)$ is the same as giving an 'explicit' description of the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ through set language, in terms of (hopefully just a few) solutions of the system.

That amounts to finding all solutions of $\mathcal{LS}(A, \mathbf{0})$.

2. **Example ($\star$).**

Determine the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

explicitly (in terms of concrete vectors in $\mathbb{R}^7$*).*

(a) First determine the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations, say, Gaussian elimination, to the augmented matrix representation of $\mathcal{LS}(A,\, \mathbf{0})$:

$$[A|\mathbf{0}] \longrightarrow \cdots\cdots\cdots \longrightarrow [A'|\mathbf{0}]$$

We find that

$$A' = \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. **Example** ($\star$).

Determine the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

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As a whole, this (4x8)-matrix is the reduced row-echelon form which is row-equivalent to the (4x8)-matrix [A|0].

We find that

$$A' = \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When this A' is singled out, it is the reduced row-echelon form which is row-equivalent to the matrix A. (This is because throughout the manipulations, the last column stays '0'.)

(b) $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 + 4x_2 + 2x_5 + x_6 - 3x_7 = 0 \\ x_3 + x_5 - 3x_6 + 5x_7 = 0 \\ x_4 + 2x_5 - 6x_6 + 6x_7 = 0 \\ 0 = 0 \end{cases}$$

The solutions of $\mathcal{LS}(A', \mathbf{0})$, and hence of $\mathcal{LS}(A, \mathbf{0})$, are given by $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$, where c_1, c_2, c_3, c_4 are arbitrary numbers, in which

$$\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This amounts to saying that for any $\mathbf{v} \in \mathbb{R}^7$, '$\mathbf{x} = \mathbf{v}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$ if and only if there exist some $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$.

(b) $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 + 4x_2 \qquad\qquad\qquad + 2x_5 + x_6 - 3x_7 = 0 \\ \qquad\qquad x_3 \qquad\quad + x_5 - 3x_6 + 5x_7 = 0 \\ \qquad\qquad\qquad x_4 + 2x_5 - 6x_6 + 6x_7 = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 = 0 \end{cases}$$

(#) is just another way of stating what you are used to write: The solutions of $\mathcal{LS}(A,0)$ are given by

$$\mathbf{x} = \begin{bmatrix} -4c_1 - 2c_2 - c_3 + 3c_4 \\ c_1 \\ -c_2 + 3c_3 - 5c_4 \\ -2c_2 + 6c_3 - 6c_4 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

where c_1, c_2, c_3, c_4 are arbitrary numbers.

(#) The solutions of $\mathcal{LS}(A', \mathbf{0})$, and hence of $\mathcal{LS}(A, \mathbf{0})$, are given by $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$, where c_1, c_2, c_3, c_4 are arbitrary numbers, in which

$$\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This amounts to saying that for any $\mathbf{v} \in \mathbb{R}^7$, '$\mathbf{x} = \mathbf{v}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$ if and only if there exist some $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$.

(c) We now apply the method of specification to present the solution set of $\mathcal{LS}(A, \mathbf{0})$ explicitly in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, through the 'selection criterion'

(†) ' *there exist some* $c_1, c_2, c_3, c_4 \in \mathbb{R}$ *such that* $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$'

in which $\mathbf{y}$ is the 'variable'.

How does the method work? Remember:

- Those and only those vectors in $\mathbb{R}^7$ which upon substitution into the symbol $\mathbf{y}$ in (†) turn it into a true statement will be collected.
- The others will be 'discarded'.

So $\mathcal{N}(A)$ is the set

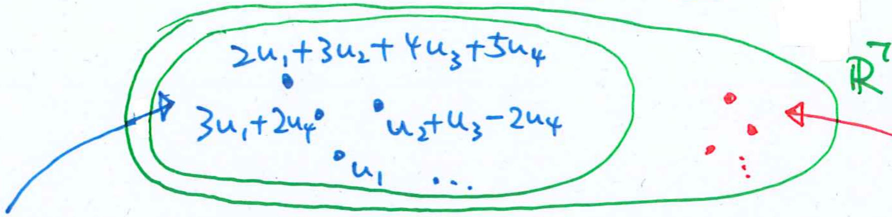
$$\left\{ \mathbf{y} \in \mathbb{R}^7 : \begin{array}{l} \text{there exist some } c_1, c_2, c_3, c_4 \in \mathbb{R} \\ \text{such that } \mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \end{array} \right\}$$

(As shorthand we present $\mathcal{N}(A)$ as $\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$.)

(c) We now apply the method of specification to present the solution set of $\mathcal{LS}(A, \mathbf{0})$ explicitly in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, through the 'selection criterion'

(†) '*there exist some* $c_1, c_2, c_3, c_4 \in \mathbb{R}$ *such that* $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$'

in which $\mathbf{y}$ is the 'variable'.



How does the method work? Remember:

- Those and only those vectors in $\mathbb{R}^7$ which upon substitution into the symbol $\mathbf{y}$ in (†) turn it into a true statement will be collected. (They are exactly all possible solutions of $\mathcal{LS}(A,0)$.)
- The others will be 'discarded'. (They are exactly all vectors in $\mathbb{R}^7$ which are not solutions of $\mathcal{LS}(A,0)$.)

So $\mathcal{N}(A)$ is the set

$$\left\{ \mathbf{y} \in \mathbb{R}^7 : \begin{array}{l} \text{there exist some } c_1, c_2, c_3, c_4 \in \mathbb{R} \\ \text{such that } \mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \end{array} \right\}$$

(As shorthand we present $\mathcal{N}(A)$ as $\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$.)

When people write 'the solution set of $\mathcal{LS}(A,0)$ is $\{c_1u_1+c_2u_2+c_3u_3+c_4u_4 \mid c_1,c_2,c_3,c_4 \in \mathbb{R}\}$', what they mean is
'the set equality $\mathcal{N}(A) = \{c_1u_1+c_2u_2+c_3u_3+c_4u_4 \mid c_1,c_2,c_3,c_4 \in \mathbb{R}\}$ holds'.
This is just a fanciful way of writing (without the use of set language)
'x is a solution of $Ax=0$ if and only if there exist some $c_1,c_2,c_3,c_4 \in \mathbb{R}$ such that $x = c_1u_1+c_2u_2+c_3u_3+c_4u_4$'.

(d) **Comment on the presentation of the manipulations.**

During the manipulation

$$[A|\mathbf{0}] \longrightarrow \cdots\cdots\cdots \longrightarrow [A'|\mathbf{0}]$$

we observe that the last column in every matrix in this sequence stays '$\mathbf{0}$'. This is expected: no matter which row-operation is applied on the zero vector, it only convert the zero vector to itself.

Hence we can actually save time (and ink) by omitting the $\mathbf{0}$'s throughout, and simply write

$$A \longrightarrow \cdots\cdots\cdots \longrightarrow A'$$

provided we remember we are apply row operations on the coefficient matrices of various homogeneous system.

3. **Examples on determining null space explicitly.**

(a) *Determine the null space of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$$

Determine the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 2 & 6 & 5 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{-1R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A'$$

The null space $\mathcal{N}(A)$ of the matrix A is the solution set of $\mathcal{LS}(A, \mathbf{0})$, and hence is the solution set of $\mathcal{LS}(A', \mathbf{0})$ as well.
Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 & & & = 0 \\ & x_2 & & = 0 \\ & & x_3 & = 0 \end{cases}$$

The only solution of $\mathcal{LS}(A, \mathbf{0})$ is given by $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Hence $\mathcal{N}(A)$ is the set $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(b) *Determine the null space of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$$

Determine the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix} \xrightarrow{-1R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{-1R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} = A'$$

The null space $\mathcal{N}(A)$ of the matrix A is the solution set of $\mathcal{LS}(A,\,\mathbf{0})$, and hence is the solution set of $\mathcal{LS}(A',\,\mathbf{0})$ as well. Note that $\mathcal{LS}(A',\,\mathbf{0})$ reads:

$$\begin{cases} x_1 & & & + & 2x_4 & = & 0 \\ & x_2 & & - & 3x_4 & = & 0 \\ & & x_3 & + & 4x_4 & = & 0 \end{cases}$$

The solutions of $\mathcal{LS}(A,\,\mathbf{0})$ are given by

$\mathbf{x} = t\mathbf{u}$, where t is an arbitrary number, in which

$$\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}.$$

Hence $\mathcal{N}(A)$ is the set $\{t\mathbf{u} \mid t \in \mathbb{R}\}$.

The null space $\mathcal{N}(A)$ of the matrix A is the solution set of $\mathcal{LS}(A, \mathbf{0})$, and hence is the solution set of $\mathcal{LS}(A', \mathbf{0})$ as well. Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 \qquad\qquad + 2x_4 = 0 \\ \quad x_2 \qquad - 3x_4 = 0 \\ \qquad\quad x_3 + 4x_4 = 0 \end{cases}$$

The solutions of $\mathcal{LS}(A, \mathbf{0})$ are given by $\mathbf{x} = t\mathbf{u}$, where t is an arbitrary number, in which

$$\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}.$$

← what we mean is: $x \in \mathbb{R}^4$ is a solution of $\mathcal{LS}(A, 0)$ if and only if there exits some $t \in \mathbb{R}$ such that $x = tu$.

Hence $\mathcal{N}(A)$ is the set $\{t\mathbf{u} \mid t \in \mathbb{R}\}$.

← This is a short-hand for

$\{x \in \mathbb{R}^4 :$ There exit some $t \in \mathbb{R}$ such that $x = tu$. $\}$

(c) *Determine the null space of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$$

Determine the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{bmatrix}$$

$$\xrightarrow{-1R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & -5 & 5 & -10 & -20 \end{bmatrix} \xrightarrow{5R_2+R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_6+R_1} \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

The null space $\mathcal{N}(A)$ of the matrix A is the solution set of $\mathcal{LS}(A,\,\mathbf{0})$, and hence is the solution set of $\mathcal{LS}(A',\,\mathbf{0})$ as well. Note that $\mathcal{LS}(A',\,\mathbf{0})$ reads:

$$\begin{cases} x_1 \phantom{{}+x_2} + 2x_3 - 3x_4 - x_5 = 0 \\ \phantom{x_1 +{}} x_2 - x_3 + 2x_4 + 4x_5 = 0 \\ \phantom{x_1 + x_2 + x_3 + x_4 + x_5 {}} 0 = 0 \end{cases}$$

The solutions of $\mathcal{LS}(A,\,\mathbf{0})$ are given by

$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, where c_1, c_2, c_3 are arbitrary numbers, in which

$$\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $\mathcal{N}(A)$ is the set $\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$.

The null space $\mathcal{N}(A)$ of the matrix A is the solution set of $\mathcal{LS}(A, \mathbf{0})$, and hence is the solution set of $\mathcal{LS}(A', \mathbf{0})$ as well. Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 \quad\quad + 2x_3 - 3x_4 - x_5 = 0 \\ \quad\quad x_2 - x_3 + 2x_4 + 4x_5 = 0 \\ \quad\quad\quad\quad\quad\quad\quad\quad 0 = 0 \end{cases}$$

The solutions of $\mathcal{LS}(A, \mathbf{0})$ are given by

$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, where c_1, c_2, c_3 are arbitrary numbers, in which

$$\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This is just $x = \begin{bmatrix} -2c_1 + 3c_2 + c_3 \\ c_1 - 2c_2 - 4c_3 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$ in another disguise.

What we mean is: $x \in \mathbb{R}^5$ is a solution of $\mathcal{LS}(A, 0)$ if and only if there exist some $c_1, c_2, c_3 \in \mathbb{R}$ such that $x = c_1u_1 + c_2u_2 + c_3u_3$.

Hence $\mathcal{N}(A)$ is the set $\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$.

This is a short-hand for $\{x \in \mathbb{R}^5 :$ there exist some $c_1, c_2, c_3 \in \mathbb{R}$ such that $x = c_1u_1 + c_2u_2 + c_3u_3 \}$.

4. **Further consideration on Example ($\star$).**

Let

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

Recall that $\mathcal{N}(A)$ is the set

$$\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\},$$

in which

$$\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) **Further question.**

What is so special about $\mathcal{N}(A)$, regarding its 'algebraic structure'?

Answer to further question.

The statements below hold:

(1) $\mathbf{0} \in \mathcal{N}(A)$.

(2) *For any* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$, *if* $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ *then* $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) *For any* $\mathbf{v} \in \mathbb{R}^7$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{v} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{v} \in \mathcal{N}(A)$.

(4) *For any* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$, *for any* $\alpha, \beta \in \mathbb{R}$, *if* $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{v} + \beta\mathbf{w} \in \mathcal{N}(A)$.

(b) **Justification of (1), (2), (3) in answer to further question.**

To apply what we see about $\mathcal{N}(A)$ in concrete terms?

(1) Note that $\mathbf{0} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3 + 0 \cdot \mathbf{u}_4$, and $0 \in \mathbb{R}$.

Then $\mathbf{0} \in \mathcal{N}(A)$.

(a) **Further question.**

What is so special about $\mathcal{N}(A)$, regarding its 'algebraic structure'?

Answer to further question.

The statements below hold:

This '0' stands for the zero vector in $\mathbb{R}^7$.

(1) $\mathbf{0} \in \mathcal{N}(A)$.

(2) *For any* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$, *if* $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ *then* $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) *For any* $\mathbf{v} \in \mathbb{R}^7$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{v} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{v} \in \mathcal{N}(A)$.

(4) *For any* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$, *for any* $\alpha, \beta \in \mathbb{R}$, *if* $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{v} + \beta\mathbf{w} \in \mathcal{N}(A)$.

(b) **Justification of (1), (2), (3) in answer to further question.**

To apply what we see about $\mathcal{N}(A)$ in concrete terms?

$\mathcal{N}(A) = \{ y \in \mathbb{R}^7 :$ there exist some $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $y = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4. \}$

(1) Note that $\mathbf{0} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3 + 0 \cdot \mathbf{u}_4$, and $0 \in \mathbb{R}$.

Then $\mathbf{0} \in \mathcal{N}(A)$.

So, given any $y \in \mathbb{R}^7$,
$y \in \mathcal{N}(A)$
if and only if
there exist some $c_1, c_2, c_3, c_4 \in \mathbb{R}$
such that
$y = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4$.

(2) Pick any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$.
Suppose $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$.

Then there exist some $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4 \in \mathbb{R}$ such that
$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ and $\mathbf{w} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + d_3\mathbf{u}_3 + d_4\mathbf{u}_4$.

Then

$$\mathbf{v} + \mathbf{w} = \cdots = (c_1 + d_1)\mathbf{u}_1 + (c_2 + d_2)\mathbf{u}_2 + (c_3 + d_3)\mathbf{u}_3 + (c_4 + d_4)\mathbf{u}_4,$$

and $c_1 + d_1, c_2 + d_2, c_3 + d_3, c_4 + d_4 \in \mathbb{R}$.

Therefore $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) Pick any $\mathbf{v} \in \mathbb{R}^7$. Pick any $\alpha \in \mathbb{R}$.
Suppose $\mathbf{v} \in \mathcal{N}(A)$.

Then there exists some $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$.

Then

$$\alpha\mathbf{v} = \cdots = (\alpha c_1)\mathbf{u}_1 + (\alpha c_2)\mathbf{u}_2 + (\alpha c_3)\mathbf{u}_3 + (\alpha c_4)\mathbf{u}_4,$$

and $\alpha c_1, \alpha c_2, \alpha c_3, \alpha c_4 \in \mathbb{R}$.

Therefore $\alpha\mathbf{v} \in \mathcal{N}(A)$.

(2) Pick any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$.
Suppose $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$. [We want to deduce $v+w \in \mathcal{N}(A)$.]

Then there exist some $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4 \in \mathbb{R}$ such that
$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ and $\mathbf{w} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + d_3\mathbf{u}_3 + d_4\mathbf{u}_4$.

Then

$$\mathbf{v} + \mathbf{w} = \cdots = (c_1 + d_1)\mathbf{u}_1 + (c_2 + d_2)\mathbf{u}_2 + (c_3 + d_3)\mathbf{u}_3 + (c_4 + d_4)\mathbf{u}_4,$$

and $c_1 + d_1, c_2 + d_2, c_3 + d_3, c_4 + d_4 \in \mathbb{R}$.

Therefore $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) Pick any $\mathbf{v} \in \mathbb{R}^7$. Pick any $\alpha \in \mathbb{R}$.
Suppose $\mathbf{v} \in \mathcal{N}(A)$. [We want to deduce $\alpha v \in \mathcal{N}(A)$.]

Then there exists some $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$.

Then

$$\alpha\mathbf{v} = \cdots = (\alpha c_1)\mathbf{u}_1 + (\alpha c_2)\mathbf{u}_2 + (\alpha c_3)\mathbf{u}_3 + (\alpha c_4)\mathbf{u}_4,$$

and $\alpha c_1, \alpha c_2, \alpha c_3, \alpha c_4 \in \mathbb{R}$.

Therefore $\alpha\mathbf{v} \in \mathcal{N}(A)$.

(c) **Another justification of (1), (2), (3) in answer to further question.**

To apply the definition of null space? (This is a better method.)

(1) Note that $A\mathbf{0} = \mathbf{0}$. Then $\mathbf{0} \in \mathcal{N}(A)$.

(2) Pick any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$.
Suppose $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$. Then $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$.
Therefore $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
Hence $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) Pick any $\mathbf{v} \in \mathbb{R}^7$. Pick any $\alpha \in \mathbb{R}$.
Suppose $\mathbf{v} \in \mathcal{N}(A)$. Then $A\mathbf{v} = \mathbf{0}$.
Therefore $A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha \cdot \mathbf{0} = \mathbf{0}$.
Hence $\alpha\mathbf{v} \in \mathcal{N}(A)$.

Remark.

This 'second justification' of (1), (2), (3) is superior to the 'first', in the sense that almost nothing about explicit features of A, apart from the fact that it has 7 columns, is involved in the mathematical argument. We may wonder the mathematical reasoning in this 'second justification' may work when A is replaced by a general matrix. It turns out to be the case.

(c) **Another justification of (1), (2), (3) in answer to further question.**

To apply the definition of null space? (This is a better method.)

By definition, $\mathcal{N}(A) = \{y \in \mathbb{R}^7 : Ay = 0_4\}$.
So, given any $y \in \mathbb{R}^7$, $y \in \mathcal{N}(A)$ if and only if $Ay = 0_4$.

This '0' stands for the zero vector in $\mathbb{R}^7$.

(1) Note that $A\mathbf{0} = \mathbf{0}$. Then $\mathbf{0} \in \mathcal{N}(A)$.

(2) Pick any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$.
Suppose $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$. Then $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$.
Therefore $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
Hence $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

These '0's stand for the zero vector in $\mathbb{R}^4$.

(3) Pick any $\mathbf{v} \in \mathbb{R}^7$. Pick any $\alpha \in \mathbb{R}$.
Suppose $\mathbf{v} \in \mathcal{N}(A)$. Then $A\mathbf{v} = \mathbf{0}$.
Therefore $A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha \cdot \mathbf{0} = \mathbf{0}$.
Hence $\alpha\mathbf{v} \in \mathcal{N}(A)$.

Remark.

This 'second justification' of (1), (2), (3) is superior to the 'first', in the sense that almost nothing about explicit features of A, apart from the fact that it has 7 columns, is involved in the mathematical argument. We may wonder the mathematical reasoning in this 'second justification' may work when A is replaced by a general matrix. It turns out to be the case.

5. **Theorem (1). (Null space of a matrix as a 'subspace'.)**

Let A be an $(m \times n)$-matrix. The statement below hold:

(1) $\mathbf{0} \in \mathcal{N}(A)$.

(2) *For any* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, *if* $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ *then* $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$.

(3) *For any* $\mathbf{v} \in \mathbb{R}^n$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{v} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{v} \in \mathcal{N}(A)$.

(4) *For any* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, *for any* $\alpha, \beta \in \mathbb{R}$, *if* $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{v} + \beta\mathbf{w} \in \mathcal{N}(A)$.

Proof. Exercise. (Extract what we did in the study of Example ($\star$).)

Remark. We can further deduce that

For any $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathbb{R}^n$, *for any* $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$, *if* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathcal{N}(A)$ *then* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k \in \mathcal{N}(A)$.

In plain words (and in terms of the notion of *linear combinations*, to be introduced later), this amounts to saying:

Every linear combination of vectors in $\mathcal{N}(A)$ *is a vector in* $\mathcal{N}(A)$.

6. **Reformulation of Theorem (1) in terms of homogeneous systems.**

Let A be an $(m \times n)$-matrix. The statement below hold:

(1) *'$\mathbf{x} = \mathbf{0}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$.*

(2) *Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Suppose '$\mathbf{x} = \mathbf{v}$', '$\mathbf{x} = \mathbf{w}$' are solutions of $\mathcal{LS}(A, \mathbf{0})$. Then '$\mathbf{x} = \mathbf{v} + \mathbf{w}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$.*

(3) *Let $\mathbf{v} \in \mathbb{R}^n$. Let $\alpha \in \mathbb{R}$. Suppose '$\mathbf{x} = \mathbf{v}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$. Then '$\mathbf{x} = \alpha\mathbf{v}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$.*

(4) *Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Let $\alpha, \beta \in \mathbb{R}$. Suppose '$\mathbf{x} = \mathbf{v}$', '$\mathbf{x} = \mathbf{w}$' are solutions of $\mathcal{LS}(A, \mathbf{0})$. Then '$\mathbf{x} = \alpha\mathbf{v} + \beta\mathbf{w}$' is a solution of $\mathcal{LS}(A, \mathbf{0})$.*