1. **Definition. (Square matrix.)**

   A matrix with the same number of rows as of columns is called a square matrix.

2. **Definition. (Non-negative powers of matrix.)**

   Let A be a square matrix.

   For each positive integer p, we define the square matrix A^p by

$$A^p = \underbrace{((\cdots(((AA)A)A)\cdots)A)A}_{p \text{ copies of } A}.$$

Remark.

We call A^2 the square of A and A^3 the cube of A et cetera.

By convention, we understand A^1 as A, and A^0 as I_n when A is a $(n \times n)$-matrix.

1. **Definition. (Square matrix.)**

A matrix with the same number of rows as of columns is called a square matrix.

2. **Definition. (Non-negative powers of matrix.)**

Let A be a square matrix.

For each positive integer p, we define the square matrix A^p by

$$A^p = \underbrace{((\cdots(((AA)A)A)\cdots)A)A}_{p \text{ copies of } A}.$$

What is it saying in plain words?

$A^2 = AA$

$A^3 = (AA)A = A^2A$

$A^4 = ((AA)A)A = A^3A$

$A^5 = A^4A$

$\vdots$

Remark.

We call A^2 the square of A and A^3 the cube of A et cetera.

By convention, we understand A^1 as A, and A^0 as I_n when A is a $(n \times n)$-matrix.

3. **Definition. (Idempotent matrices.)**

Suppose A is a square matrix.

Then A is said to be idempotent if and only if $A^2 = A$.

4. **Examples on idempotent matrices.**

(a) The $(n \times n)$-zero matrix is idempotent.

Reason: Note that ${\mathcal{O}_{n\times n}}^2 = \mathcal{O}_{n\times n}$.

(b) The $(n \times n)$-identity matrix is idempotent.

Reason: Note that ${I_n}^2 = I_n$.

(c) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We have $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$.

Then A is idempotent.

(d) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. We have $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A$.

Then A is idempotent.

Remark.

By definition, given that A is an $(n\times n)$-idempotent matrix, it will happen that $A(A-I_n) = A^2 - A = \mathcal{O}_{n\times n}$.

But as suggested by the examples above, it does not follow that $A = \mathcal{O}_{n\times n}$ or $A = I_n$.

> The point in this passage is to explain what is meant by 'the matrix A is idempotent'.

> It is under such an assumption that whether a matrix is idempotent is considered.

3. Definition. (Idempotent matrices.)

Suppose A is a square matrix.

Then A is said to be idempotent if and only if $A^2 = A$.

> This is the 'signpost' for where the 'defining condition' begins.

> This is the 'defining condition', which explains what 'the matrix A is idempotent' means in terms of something previously defined, namely, 'square of matrices':
> - when $A^2 = A$, A is idempotent.
> - when $A^2 \neq A$, A is not idempotent
>
> of course, by logic:
> - when A is idempotent, $A^2 = A$.
> - when A is not idempotent, $A^2 \neq A$.

4. Examples on idempotent matrices.

(a) The $(n \times n)$-zero matrix is idempotent.

Reason: Note that $\mathcal{O}_{n\times n}{}^2 = \mathcal{O}_{n\times n}$.

(b) The $(n \times n)$-identity matrix is idempotent.

Reason: Note that $I_n{}^2 = I_n$.

(c) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We have $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$.

Then A is idempotent.

(d) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. We have $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A$.

Then A is idempotent.

> Can you name two more (2×2)-matrices which are idempotent? (Check your answer.)

Remark.

> Digression from (c), (d) above.

By definition, given that A is an $(n\times n)$-idempotent matrix, it will happen that $A(A - I_n) = A^2 - A = \mathcal{O}_{n\times n}$.

But as suggested by the examples above, it does not follow that $A = \mathcal{O}_{n\times n}$ or $A = I_n$.

Non-examples.

(a) Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

We have $B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq B$.

Then B is not idempotent.

(b) Let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

We have $B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq B$.

Then B is not idempotent.

Non-examples.

(a) Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

We have $B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq B$.

Then B is not idempotent.

(b) Let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

We have $B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq B$.

Then B is not idempotent.

Can you name two more (2×2)-matrices which are not idempotent? (Check your answer.)

5. **Definition. (Nilpotent matrices.)**

Suppose A is a square matrix.

Then A is said to be nilpotent if and only if there is some positive integer p so that $A^p = \mathcal{O}$.

6. **Examples on nilpotent matrices.**

(a) The $(n \times n)$-zero matrix is nilpotent.

Reason: Note that ${\mathcal{O}_{n\times n}}^1 = \mathcal{O}_{n\times n}$.

(b) Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We have

$$A^2 = \cdots = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \cdots = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^4 = \cdots = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathcal{O}_{4\times 4}.$$

Therefore A is nil-potent.

5. **Definition. (Nilpotent matrices.)**

Suppose A is a square matrix.

Then A is said to be nilpotent if and only if there is some positive integer p so that $A^p = \mathcal{O}$.

Somewhere amongst the positive integers, there is one positive integer, which for convenience we label as p, satisfying $A^p = \mathcal{O}$.
The 'value' of such a p depends on what A is.

6. **Examples on nilpotent matrices.**

(a) The $(n \times n)$-zero matrix is nilpotent.

Reason: Note that $\mathcal{O}_{n\times n}{}^1 = \mathcal{O}_{n\times n}$.

There exists some positive integer p, namely $p=1$, such that $\mathcal{O}_{n\times n}{}^p = \mathcal{O}_{n\times n}$

(b) Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We have

$$A^2 = \cdots = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \cdots = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^4 = \cdots = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathcal{O}_{4\times 4}.$$

Therefore A is nil-potent.

There exists some positive integer p, namely $p=4$, such that $A^p = \mathcal{O}_{4\times 4}$.

(c) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$. We have $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathcal{O}_{3\times 3}$.

Therefore A is nil-potent.

Remark.

It is possible for some non-zero matrix to be 'self-multiplied' for sufficiently many times to result in the zero matrix.

Non-examples.

(a) The $(n \times n)$-identity matrix is not nilpotent.

Reason: Note that ${I_n}^2 = I_n$.

Then for each positive integer p, we have ${I_n}^p = {I_n}^{p-1} = \cdots = {I_n}^2 = I_n \neq \mathcal{O}_{n\times n}$.

(b) Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

We have $B^2 = \cdots = B$.

Then for each positive integer p, we have $B^p = B \neq \mathcal{O}$.

Then B is not nilpotent.

(c) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$. We have $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathcal{O}_{3\times 3}$.

There exits some positive integer p, namely $p = 3$, such that $A^p = \mathcal{O}_{3\times 3}$.

Therefore A is nil-potent.

Can you name two more (3x3)-matrices which are nilpotent? How about (4x4)-matrices? (Check your answer.)

Remark.

It is possible for some non-zero matrix to be 'self-multiplied' for sufficiently many times to result in the zero matrix.

Non-examples.

(a) The $(n \times n)$-identity matrix is not nilpotent.

Re-formulation: 'For each positive integer p, $I_n^p \neq \mathcal{O}_{n\times n}$.'

Reason: Note that ${I_n}^2 = I_n$.

Then for each positive integer p, we have ${I_n}^p = {I_n}^{p-1} = \cdots = {I_n}^2 = I_n \neq \mathcal{O}_{n\times n}$.

(b) Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

[We verify 'B is not nilpotent'.]

Re-formulation of 'B is not nilpotent': 'For each positive integer p, $B^p \neq \mathcal{O}_{3\times 3}$.'

We have $B^2 = \cdots = B$.

Then for each positive integer p, we have $B^p = B \neq \mathcal{O}$.

Then B is not nilpotent.

7. **Definition. (Commuting matrices.)**

Suppose A, B are $(n \times n)$-square matrices. Then A, B are said to commute with each other if and only if $AB = BA$. We can also say that A, B are a pair of commuting matrices.

8. **Examples on commuting matrices.**

(a) The $(n \times n)$-zero matrix commute with every $(n \times n)$-square matrix.

(b) The $(n \times n)$-identity matrix commute with every $(n \times n)$-square matrix.

(c) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$. We have

$$AB = \cdots = \begin{bmatrix} 12 & 0 \\ 0 & 15 \end{bmatrix}, \quad BA = \cdots = \begin{bmatrix} 12 & 0 \\ 0 & 15 \end{bmatrix}.$$

Then $AB = BA$. Therefore A, B commute with each other.

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. We have

$$AB = \cdots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad BA = \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $AB = BA$. Therefore A, B commute with each other.

7. **Definition. (Commuting matrices.)**
Suppose A, B are $(n \times n)$-square matrices. Then A, B are said to commute with each other if and only if $AB = BA$. We can also say that A, B are a pair of commuting matrices.

Name to be introduce for convenience in communications.

8. **Examples on commuting matrices.**

(a) The $(n \times n)$-zero matrix commute with every $(n \times n)$-square matrix.

(b) The $(n \times n)$-identity matrix commute with every $(n \times n)$-square matrix.

(c) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$. We have

$$AB = \cdots = \begin{bmatrix} 12 & 0 \\ 0 & 15 \end{bmatrix}, \quad BA = \cdots = \begin{bmatrix} 12 & 0 \\ 0 & 15 \end{bmatrix}.$$

Then $AB = BA$. Therefore A, B commute with each other.

Can you generalize this example to other (2×2)-matrices? How about (3×3)-matrices and (4×4)-matrices? Check your answer.

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. We have

$$AB = \cdots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad BA = \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $AB = BA$. Therefore A, B commute with each other.

Non-examples.

(a) Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We have

$$AB = \cdots = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad BA = \cdots = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $AB \neq BA$. Therefore A, B do not commute.

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. We have

$$AB = \cdots = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad BA = \cdots = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then $AB \neq BA$. Therefore A, B do not commute.

Remark.

As suggested by these non-examples on commuting matrices, there is no such thing as the 'Law of Commutativity for matrix multiplication'.

Formally speaking, the statement below is false:

Let n be an integer greater than 1. Suppose A, B are $(n \times n)$-matrices. Then $AB = BA$.

There is something non-trivial for a pair of square matrices to commute.

Non-examples.

(a) Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We have

$$AB = \cdots = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad BA = \cdots = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $AB \neq BA$. Therefore A, B do not commute.

Can you give some other examples from (2×2)-matrices?

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. We have

$$AB = \cdots = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad BA = \cdots = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then $AB \neq BA$. Therefore A, B do not commute.

Remark.

Digression from the non-examples.

As suggested by these non-examples on commuting matrices, there is no such thing as the 'Law of Commutativity for matrix multiplication'.

Formally speaking, the statement below is false:

Let n be an integer greater than 1. Suppose A, B are $(n \times n)$-matrices. Then $AB = BA$.

There is something non-trivial for a pair of square matrices to commute.

↑ This is the point in the definition for the notion of commuting matrices.

9. **Definition. (Lie product for square matrices.)**

Let A, B be $(n \times n)$-square matrices with real entries.

The $(n \times n)$-square matrix $AB - BA$ is called the Lie product of A, B, and is denoted by $[A, B]$.

Remark. $[A, B]$ 'measures' how far AB and BA differ from each other.

10. **Examples on Lie product.**

Let $J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

We have $JK = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $KJ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then $[J, K] = JK - KJ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = L$.

Similarly $[K, L] = J$, and $[L, J] = K$. (Fill in the detail.)

We also have $[I_n, J] = [I_n, K] = [I_n, L] = \mathcal{O}_{3\times 3}$.

This is another format for definitions: we give a name and a symbol for a certain type of objects.

9. **Definition. (Lie product for square matrices.)**
Let A, B be $(n \times n)$-square matrices with real entries.
The $(n \times n)$-square matrix $AB - BA$ is called the Lie product of A, B, and is denoted by $[A, B]$.

Remark. $[A, B]$ 'measures' how far AB and BA differ from each other.

10. **Examples on Lie product.**

Let $J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

We have $JK = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $KJ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then $[J, K] = JK - KJ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = L$.

Similarly $[K, L] = J$, and $[L, J] = K$. (Fill in the detail.)

We also have $[I_n, J] = [I_n, K] = [I_n, L] = \mathcal{O}_{3\times 3}$.

11. **Definition. (Invertible matrices.)**

Let A be an $(n \times n)$-square matrix.

(a) *Suppose B is a $(n \times n)$-square matrix. Further suppose $BA = I_n$ and $AB = I_n$. Then we say B is a matrix inverse of A.*

(b) *A is said to be invertible if and only if A has a matrix inverse.*

12. **Examples on invertible matrices.**

(a) The identity matrix is invertible.

(b) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

We have $BA = \cdots = I_2$ and $AB = \cdots = I_2$.

Then A is invertible, and B is a matrix inverse of A.

(c) Let $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \end{bmatrix}$, and $B = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

We have $BA = \cdots = I_3$ and $AB = \cdots = I_3$.

Then A is invertible, and B is a matrix inverse of A.

11. Definition. (Invertible matrices.)

Let A be an $(n \times n)$-square matrix.

(a) *Suppose B is a $(n \times n)$-square matrix. Further suppose $BA = I_n$ and $AB = I_n$. Then we say B is a matrix inverse of A.*

(b) *A is said to be invertible if and only if A has a matrix inverse.*

In (a), we are explaining what 'B is a matrix inverse of A' means, in terms of something introduced previously:
B is a matrix inverse of A exactly when (B is a square matrix and) $BA = I_n$ and $AB = I_n$.

In (b), we are explaining what 'A is invertible' in terms of what we have introduced in (a).

12. Examples on invertible matrices.

(a) The identity matrix is invertible.

Reason: Write $B = I_n$. $BI_n = I_n^2 = I_n$. Also $I_nB = I_n^2 = I_n$.

(b) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

We have $BA = \cdots = I_2$ and $AB = \cdots = I_2$.

Then A is invertible, and B is a matrix inverse of A.

(c) Let $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \end{bmatrix}$, and $B = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

We have $BA = \cdots = I_3$ and $AB = \cdots = I_3$.

Then A is invertible, and B is a matrix inverse of A.

Non-examples.

(a) The $(n \times n)$-zero matrix is not invertible.

Reason: For any $(n \times n)$-square matrix B, it happens that $\mathcal{O}_{n\times n}B = \mathcal{O}_{n\times n} \neq I_n$.

(b) Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Pick any (3×3)-matrix B. Denote the (i, j)-th entry of B by b_{ij}. (So $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.)

We have

$$AB = \cdots = \begin{bmatrix} b_{21} + b_{31} & b_{22} + b_{32} & b_{23} + b_{33} \\ b_{31} & b_{32} & b_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

The $(3, 3)$-th entry of AB is 0.

Therefore $AB \neq I_3$. (This happens no matter what B is in the first place.)

Hence A is not invertible.

An $(n\times n)$-matrix A is not invertible exactly when A does not have a matrix inverse.

Re-formulation: No matter which $(n\times n)$-matrix B is selected, it happens that at least one of '$BA=I_n$', '$AB=I_n$' fails to hold.

Non-examples.

(a) The $(n \times n)$-zero matrix is not invertible.

Reason: For any $(n \times n)$-square matrix B, it happens that $\mathcal{O}_{n\times n}B = \mathcal{O}_{n\times n} \neq I_n$.

(b) Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

[We verify: No matter which (3×3)-matrix B is selected, it happens that '$AB=I_3$' fails to hold.]

Pick any (3×3)-matrix B. Denote the (i, j)-th entry of B by b_{ij}. (So $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.)

We have

$$AB = \cdots = \begin{bmatrix} b_{21} + b_{31} & b_{22} + b_{32} & b_{23} + b_{33} \\ b_{31} & b_{32} & b_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

← This has no chance to be I_3.

The $(3, 3)$-th entry of AB is 0.

Therefore $AB \neq I_3$. (This happens no matter what B is in the first place.)

Hence A is not invertible.

(c) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Pick any (2×2)-matrix B. Denote the (i, j)-th entry of B by b_{ij}. (So $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.)

We have

$$AB = \cdots = \begin{bmatrix} b_{11} + 2b_{21} & b_{12} + 2b_{22} \\ 2b_{11} + 4b_{21} & 2b_{12} + 4b_{22} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 2\alpha & 2\beta \end{bmatrix}$$

in which $\alpha = b_{11} + 2b_{21}$ and $\beta = b_{12} + 2b_{22}$.

Then the entries in the first column of AB are all zero, or all non-zero. Therefore $AB \neq I_2$. (This happens no matter what B is in the first place.)

Hence A is not invertible.

Remark.

As suggested by these non-examples on matrix inverse, there is no such thing as the 'Law of Existence of Inverse for matrix multiplication'.

Formally speaking, the statement below is false:

> *Let n be an integer greater than 1. Suppose A is a non-zero $(n \times n)$-square matrix. Then there exists some $(n \times n)$-square matrix B such that $BA = I_n$ and $AB = I_n$.*

There is something non-trivial for a square matrix to be invertible.

(c) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

We verify: No matter which (2×2)-matrix is selected, it happens that '$AB = I_2$' fails to hold.

Pick any (2×2)-matrix B. Denote the (i, j)-th entry of B by b_{ij}. (So $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.)

We have

$$AB = \cdots = \begin{bmatrix} b_{11} + 2b_{21} & b_{12} + 2b_{22} \\ 2b_{11} + 4b_{21} & 2b_{12} + 4b_{22} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 2\alpha & 2\beta \end{bmatrix}$$

in which $\alpha = b_{11} + 2b_{21}$ and $\beta = b_{12} + 2b_{22}$.

Then the entries in the first column of AB are all zero, or all non-zero. Therefore $AB \neq I_2$. (This happens no matter what B is in the first place.)

Hence A is not invertible.

Remark.

As suggested by these non-examples on matrix inverse, there is no such thing as the 'Law of Existence of Inverse for matrix multiplication'.

Formally speaking, the statement below is false:

> *Let n be an integer greater than 1. Suppose A is a non-zero $(n \times n)$-square matrix. Then there exists some $(n \times n)$-square matrix B such that $BA = I_n$ and $AB = I_n$.*

There is something non-trivial for a square matrix to be invertible.

13. **Definition. (Transpose.)**

Let A be an $(m \times n)$-matrix, whose (i, j)-th entry is denoted by a_{ij}.

The $(n \times m)$-matrix whose (k, ℓ)-th entry is given by $a_{\ell k}$ is called the transpose of A, and is denoted by A^t.

$$\text{(So } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \text{ where as } A^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & a_{23} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.)$$

13. **Definition. (Transpose.)**

Let A be an $(m \times n)$-matrix, whose (i, j)-th entry is denoted by a_{ij}.

The $(n \times m)$-matrix whose (k, ℓ)-th entry is given by $a_{\ell k}$ is called the transpose of A, and is denoted by A^t.

$$(\text{So } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \text{ where as } A^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & a_{23} & \cdots & a_{m3} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.)$$

i-th row of A ⇝ i-th column of A^t

j-th column of A ⇝ j-th row of A

14. **Examples on transpose.**

Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}$.

Then $A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B^t = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$ and $C^t = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

(a) Note that $A + B = \begin{bmatrix} 2 & 5 & 3 \\ 2 & 2 & 3 \end{bmatrix}$. Then $(A + B)^t = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$

We have $A^t + B^t = \cdots = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$. So $(A + B)^t = A^t + B^t$ (in this example).

(b) Note that $AC = \cdots = \begin{bmatrix} 4 & 13 \\ 2 & 7 \end{bmatrix}$. Then $(AC)^t == \begin{bmatrix} 4 & 2 \\ 13 & 7 \end{bmatrix}$

We have $C^t A^t = \cdots = \begin{bmatrix} 4 & 2 \\ 13 & 7 \end{bmatrix}$. So $(AC)^t = C^t A^t$ (in this example).

15. **Definition. (Symmetric matrix and Skew-symmetric matrix.)**

Let A be an $(n \times n)$-square matrix.

(a) *A is said to be symmetric if $A^t = A$.*

(b) *A is said to be skew-symmetric if $A^t = -A$.*

16. **Examples and non-examples on symmetric matrices and skew-symmetric matrices.**

(a) The $(n \times n)$-zero matrix is a symmetric matrix. It is also a skew-symmetric matrix.

(b) The identity matrix is a symmetric matrix. It is not skew-symmetric.

(c) Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 6 \end{bmatrix}$.

Note that $A^t = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 6 \end{bmatrix} = A$. Then A is symmetric.

Note that $A^t \neq -A$. Then A is not skew-symmetric.

(d) Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$.

Note that $A^t = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -A$. Then A is skew-symmetric.

Note that $A^t \neq A$. Then A is not symmetric.

(e) Let $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Note that $B^t = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

We have $B^t \neq B$. Then B is not symmetric.

We have $B^t \neq -B$. Then B is not skew-symmetric.

(f) Let $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Note that $B^t = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

We have $B^t \neq B$. Then B is not symmetric.

We have $B^t \neq -B$. Then B is not skew-symmetric.

17. **Definition. (Orthogonal matrix.)**

Suppose A be an $(n \times n)$-square matrix.

Then A is said to be orthogonal if $AA^t = I_n$ and $A^tA = I_n$.

Remark. By definition, an orthogonal matrix is invertible, and its matrix inverse is its transpose.

18. **Examples on orthogonal matrices.**

(a) The identity matrix is an orthogonal matrix.

(b) Let θ be a real number, and $A_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$, $B_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

Note that ${A_\theta}^t = B_\theta$.

We have $A_\theta {A_\theta}^t = A_\theta B_\theta = \cdots = I_2$ and ${A_\theta}^t A_\theta = ... = I_2$.

Then A_θ is an orthogonal matrix.

Similarly, we deduce that B_θ is an orthogonal matrix.

(In fact, every (2×2)-orthogonal matrix is given by A_θ or B_θ for some real number θ.)

(c) Let $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \end{bmatrix}$.

We have $A^t = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

Then $AA^t = \cdots = I_3$ and $A^tA = \cdots = I_3$.

Therefore A is an orthogonal matrix.

Non-examples.

(a) Let $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. We have $B^t = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $BB^t = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Note that $BB^t \neq I_2$. Then B is not an orthogonal matrix.

(b) Let $B = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$. We have $B^t = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$. Then $BB^t = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$.

Note that $BB^t \neq I_2$. Then B is not an orthogonal matrix.