MATH1030 Vector presentation and matrix presentation of a system of linear equations and its solutions

1. **Miscellaneous presentations of systems of linear equations, introduced via an example.**

   Consider the system

$$(S): \left\{ \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & x_4 & = & 7 \\ x_1 & + & x_2 & + & x_3 & - & x_4 & = & 3 \\ 3x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 1 \end{array} \right. .$$

   (a) We write all the givens in the system (S) explicitly:

$$(S'): \quad \left\{ \begin{array}{rcrcrcrcr} 1x_1 & + & 2x_2 & + & 0x_3 & + & 1x_4 & = & 7 \\ 1x_1 & + & 1x_2 & + & 1x_3 & + & (-1)x_4 & = & 3 \\ 3x_1 & + & 1x_2 & + & 5x_3 & + & (-7)x_4 & = & 1 \end{array} \right.$$

   (b) With (S'), we form the augmented matrix representation of (S), which is the matrix

$$C = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{array} \right].$$

   (c) According to the definition of equality for matrices and vectors, (S') can be re-formulated as the equation for vectors

$$(S''): \quad \left[\begin{array}{rcrcrcr} 1x_1 & + & 2x_2 & + & 0x_3 & + & 1x_4 \\ 1x_1 & + & 1x_2 & + & 1x_3 & + & (-1)x_4 \\ 3x_1 & + & 1x_2 & + & 5x_3 & + & (-7)x_4 \end{array} \right] = \left[\begin{array}{c} 7 \\ 3 \\ 1 \end{array} \right]$$

   with the unknown x_1, x_2, x_3, x_4 placed in the 'left-hand side' of (S'').

   (d) According to the definition of addition and scalar multiplication for vectors, (S'') can be further re-formulated as the equation

$$(S'''): \quad x_1 \left[\begin{array}{c} 1 \\ 1 \\ 3 \end{array} \right] + x_2 \left[\begin{array}{c} 2 \\ 1 \\ 1 \end{array} \right] + x_3 \left[\begin{array}{c} 0 \\ 1 \\ 5 \end{array} \right] + x_4 \left[\begin{array}{c} 1 \\ -1 \\ -7 \end{array} \right] = \left[\begin{array}{c} 7 \\ 3 \\ 1 \end{array} \right]$$

   with unknown scalars x_1, x_2, x_3, x_4.

   (S''') is called the vector presentation of the system (S).

   (e) According to the definition of matrix multiplication, (S'') can also be re-formulated as the equation as

$$(S''''): \quad \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right] = \left[\begin{array}{c} 7 \\ 3 \\ 1 \end{array} \right]$$

   with unknown scalars x_1, x_2, x_3, x_4.

   (f) Write

$$A = \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{array} \right], \quad \mathbf{b} = \left[\begin{array}{c} 7 \\ 3 \\ 1 \end{array} \right], \quad \mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right].$$

   Then (S'''') reads as $A\mathbf{x} = \mathbf{b}$. This is called the matrix representation of the system (S), with matrix of coefficients A, vector of constants $\mathbf{b}$ and unknown vector $\mathbf{x}$.

   (g) In terms of $A, \mathbf{b}$ The augmented matrix representation C of the system (S) is given by $C = [\ A \mid \mathbf{b}\ ]$.

   (h) The 'given' vectors $\left[\begin{array}{c} 1 \\ 1 \\ 3 \end{array} \right], \left[\begin{array}{c} 2 \\ 1 \\ 1 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \\ 5 \end{array} \right], \left[\begin{array}{c} 1 \\ -1 \\ 7 \end{array} \right]$, involved in the left-hand side of the vector presentation of (S), which is

$$(S'''): \quad x_1 \left[\begin{array}{c} 1 \\ 1 \\ 3 \end{array} \right] + x_2 \left[\begin{array}{c} 2 \\ 1 \\ 1 \end{array} \right] + x_3 \left[\begin{array}{c} 0 \\ 1 \\ 5 \end{array} \right] + x_4 \left[\begin{array}{c} 1 \\ -1 \\ -7 \end{array} \right] = \left[\begin{array}{c} 7 \\ 3 \\ 1 \end{array} \right]$$

   are simply the respective columns of the matrix A, from left to right.

2. **Definition. (Matrix presentation and vector presentation of a system of linear equations.)**

   Consider the system of m linear equations with n unknowns

$$(S): \quad \left\{ \begin{array}{rcrcccrcl} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \right.$$

in which the a_{ij}'s, b_i's are the given's and the x_j's are the unknowns.

Let A be the $(m \times n)$-matrix given by $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, and $\mathbf{b}$ be the vector with m entries given by $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

(a) Write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

The equation $A\mathbf{x} = \mathbf{b}$ with unknown vector $\mathbf{x}$ in $\mathbb{R}^n$ is called the matrix representation of the system (S).

The matrix A is called the matrix of coefficients of the system (S) and the vector $\mathbf{b}$ is called the vector of constants of the system (S).

When we write $\mathcal{LS}(A, \mathbf{b})$, we are referring to this system (S).

(b) Let $\mathbf{u}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, ..., $\mathbf{u}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$. (So $A = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n\ ]$.)

The equation $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n = \mathbf{b}$ with unknown $x_1, x_2, \cdots, x_n$ is called the vector representation of the system $\mathcal{LS}(A, \mathbf{b})$.

Remark.

(a) This definition makes sense by virtue of Lemma (1), which tells us that the system (S), the simultaneous equations arising from the equation $A\mathbf{x} = \mathbf{b}$, and the the simultaneous equations arising from $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n = \mathbf{b}$ are all equivalent to each other as equations.

(b) The augmented matrix representation of $\mathcal{LS}(A, \mathbf{b})$ is the $(m \times (n+1))$-matrix given by

$$[\ A \mid \mathbf{b}\ ] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

3. **Lemma (1).**

Let a_{ij} be real numbers for each $i = 1, \cdots, m$ and for each $j = 1, \cdots, n$. Let b_k be real numbers for each $k = 1, \cdots, m$. Let t_ℓ be real numbers for each $\ell = 1, \cdots, n$

Define $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$.

Denote the j-th column of A by $\mathbf{u}_j$ for each j. (So $A = [\ \mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n\ ]$)

The statements below are logically equivalent in the sense that when one of them holds, every one of them holds. (So they are true together, or false together.)

(a) $\left\{\begin{array}{ccccccccc} a_{11}t_1 & + & a_{12}t_2 & + & \cdots & + & a_{1n}t_n & = & b_1 \\ a_{21}t_1 & + & a_{22}t_2 & + & \cdots & + & a_{2n}t_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m1}t_1 & + & a_{m2}t_2 & + & \cdots & + & a_{mn}t_n & = & b_m \end{array}\right.$.

(b) $t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_n\mathbf{u}_n = \mathbf{b}$.

(c) $A\mathbf{t} = \mathbf{b}$.

4. **Examples on various representations and presentations of systems of linear equations.**

(a) Consider the system (S) : $\left\{\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array}\right.$.

The matrix presentation of the system (S) is given by

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

The augmented matrix representation of (S) is $[\ A \mid \mathbf{b}\ ]$, in which

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

are respectively the coefficient matrix and the vector of constant of the system (S).
The vector presentation of the system (S) is given by

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

(b) Consider the system (S) : $\left\{ \begin{array}{rcrcrcrcrcr} & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \\ -2x_1 & - & x_2 & - & 3x_3 & + & 3x_4 & + & x_5 & = & 3 \end{array} \right.$.

The matrix presentation of the system (S) is given by

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ -2 & -1 & -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}.$$

The augmented matrix representation of (S) is $[\ A \mid \mathbf{b}\ ]$, in which

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ -2 & -1 & -3 & 3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

are respectively the coefficient matrix and the vector of constant of the system (S).
The vector presentation of the system (S) is given by

$$x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}.$$

(c) Consider the system (S) : $\left\{ \begin{array}{rcrcrcrcrcrcr} & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 \\ -x_1 & + & 2x_2 & + & x_3 & - & x_4 & & & - & 2x_6 & = & 0 \\ 2x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & + & 2x_5 & + & x_6 & = & 5 \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 \end{array} \right.$.

The matrix presentation of the system (S) is given by

$$\begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 \\ -1 & 2 & 1 & -1 & 0 & -2 \\ 2 & -4 & -1 & 3 & 2 & 1 \\ 3 & -6 & -1 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}.$$

The augmented matrix representation of (S) is $[\ A \mid \mathbf{b}\ ]$, in which

$$A = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 \\ -1 & 2 & 1 & -1 & 0 & -2 \\ 2 & -4 & -1 & 3 & 2 & 1 \\ 3 & -6 & -1 & 5 & 4 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}$$

are respectively the coefficient matrix and the vector of constant of the system (S).
The vector presentation of the system (S) is given by

$$x_1 \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ -4 \\ -6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 3 \\ 5 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ 2 \\ 4 \end{bmatrix} + x_6 \begin{bmatrix} -7 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}.$$

5. **Vector presentation of solutions of systems of linear equations, introduced via an example.**

Consider the system

$$(S): \left\{ \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & x_4 & = & 7 \\ x_1 & + & x_2 & + & x_3 & - & x_4 & = & 3 \\ 3x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 1 \end{array} \right. .$$

(a) After some work, we find that

(†) the solutions of (S) are described by $(x_1, x_2, x_3, x_4) = (-1-2s+3t, 4+s-2t, s, t)$ where s, t are arbitrary numbers.

(b) Invoking the 'identification' between vectors with four entries and points who are the 'arrowheads' of the vectors, we may present (†) as:

(†′) the solutions of (S) are described by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1-2s+3t \\ 4+s-2t \\ s \\ t \end{bmatrix}$ where s, t are arbitrary numbers.

We may further replace the symbols $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ by $\mathbf{x}$ when there is no ambiguity as to the meaning of the symbol $\mathbf{x}$ in the context of the discussion.

(c) For some reason which will become apparent later on in the course, it is actually best to exploit vector addition and scalar multiplication to 'split' the reference to the 'arbitrary numbers s, t', and to present (†) as:

(†″) the solutions of (S) are described by $\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t are arbitrary numbers.

6. **Examples on Vector presentation of solutions of systems of linear equations.**

(a) Consider the system $(S): \left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array} \right. .$

After some work, we find that

(†) the solutions of (S) are described by $(x_1, x_2, x_3) = (2, -3, 4)$.

We may present (†) as:

(†″) the solution of (S) is given by $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

(b) Consider the system $(S): \left\{ \begin{array}{rcrcrcrcrcr} & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \\ -2x_1 & - & x_2 & - & 3x_3 & + & 3x_4 & + & x_5 & = & 3 \end{array} \right. .$

After some work, we find that

(†) the solutions of (S) are described by $(x_1, x_2, x_3, x_4, x_5) = (10-s-t, -8-s, s, 5-t, t)$ where s, t are arbitrary numbers.

We may present (†) as:

(†′) the solution of (S) is given by $\mathbf{x} = \begin{bmatrix} 10-s-t \\ -8-s \\ s \\ 5-t \\ t \end{bmatrix}$ where s, t are arbitrary numbers.

Or further as:

(†″) the solution of (S) is given by $\mathbf{x} = \begin{bmatrix} 10 \\ -8 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ where s, t are arbitrary numbers.

(c) Consider the system $(S): \left\{ \begin{array}{rcrcrcrcrcrcr} & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 \\ -x_1 & + & 2x_2 & + & x_3 & - & x_4 & & & - & 2x_6 & = & 0 \\ 2x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & + & 2x_5 & + & x_6 & = & 5 \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 \end{array} \right. .$

After some work, we find that

(†) the solutions of (S) are described by $(x_1, x_2, x_3, x_4, x_5, x_6) = (1+2r-t, r, 3-s+2t, 2-s+t, s, t)$, where r, s, t are arbitrary numbers.

We may present (†) as:

$$(\dagger')\ \text{ the solution of } (S) \text{ is given by } \mathbf{x} = \begin{bmatrix} 1+2r-t \\ r \\ 3-s+2t \\ 2-s+t \\ s \\ t \end{bmatrix} \text{ where } r, s, t \text{ are arbitrary numbers.}$$

Or further as:

$$(\dagger'')\ \text{ the solution of } (S) \text{ is given by } \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ where } r, s, t \text{ are arbitrary numbers.}$$