1. Recall the definitions on the notion of matrices:

   A $(p \times q)$-rectangular array

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pq} \end{bmatrix}$$

   in which the c_{ij}'s are real numbers is called a $(p \times q)$-matrix with real entries, with p rows and q columns.
   Suppose We denote this matrix by C.
   For each fixed $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$, we refer to the number c_{ij} as the (i, j)-th entry of C.
   For each fixed $k = 1, 2, \cdots, p$, we refer to the array

$$\begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kq} \end{bmatrix},$$

   as the k-th row of C.
   For each fixed $\ell = 1, 2, \cdots, q$, we refer to the array

$$\begin{bmatrix} c_{1\ell} \\ c_{2\ell} \\ \vdots \\ c_{q\ell} \end{bmatrix},$$

   as the ℓ-th column of C.

2. **Definition. (Equality for matrices.)**

   Let B, C be $(p \times q)$-matrices with real entries.

   For each i, j, denote the (i, j)-th entry of B by b_{ij}, and the (i, j)-th entry of C by c_{ij}.

   B is said to be equal to C as matrices if and only if $b_{ij} = c_{ij}$ for each i, j.

3. **Definition. (Addition for matrices.)**

   Let B, C be $(p \times q)$-matrices with real entries.

   For each i, j, denote the (i, j)-th entry of B by b_{ij}, and the (i, j)-th entry of C by c_{ij}.

   The sum of B, C is defined to be the $(p \times q)$-matrix whose (i, j)-th entry is $b_{ij} + c_{ij}$ for each i, j. It is denoted by $B + C$.

   Remark. In symbols, this definition says:

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pq} \end{bmatrix} = \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} & \cdots & b_{1q} + c_{1q} \\ b_{21} + c_{21} & b_{22} + c_{22} & \cdots & b_{2q} + c_{2q} \\ \vdots & \vdots & & \vdots \\ b_{p1} + c_{p1} & b_{p2} + c_{p2} & \cdots & b_{pq} + c_{pq} \end{bmatrix}$$

4. **Definition. (Scalar multiplication for matrices.)**

   Let C be $(p \times q)$-matrices with real entries.

   For each i, j, denote the (i, j)-th entry of C by c_{ij}.

   Let α be a real number.

   The scalar multiple of C by α is defined to be the $(p \times q)$-matrix whose (i, j)-th entry is αc_{ij} for each i, j. It is denoted by αC.

   Remark. In symbols, this definition says:

$$\alpha \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pq} \end{bmatrix} = \begin{bmatrix} \alpha c_{11} & \alpha c_{12} & \cdots & \alpha c_{1q} \\ \alpha c_{21} & \alpha c_{22} & \cdots & \alpha c_{2q} \\ \vdots & \vdots & & \vdots \\ \alpha c_{p1} & \alpha c_{p2} & \cdots & \alpha c_{pq} \end{bmatrix}$$

5. **Examples.**

   (a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 5 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 6 \\ 9 & 8 & 7 \end{bmatrix}$.

(b) $3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$.

6. **Definition. (Subtraction for matrices.)**

   Let B, C be $(p \times q)$-matrices with real entries.

   The difference of B from C is defined to be the $(p \times q)$-matrix $B + (-1)C$. It is denoted by $B - C$.

   Remark. In symbols, this definition says that if

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pq} \end{bmatrix}$$

then

$$B - C = \begin{bmatrix} b_{11} - c_{11} & b_{12} - c_{12} & \cdots & b_{1q} - c_{1q} \\ b_{21} - c_{21} & b_{22} - c_{22} & \cdots & b_{2q} - c_{2q} \\ \vdots & \vdots & & \vdots \\ b_{p1} - c_{p1} & b_{p2} - c_{p2} & \cdots & b_{pq} - c_{pq} \end{bmatrix}.$$

7. **Definition. (Zero matrix.)**

   The $(p \times q)$-matrix whose entries are all 0 is called the zero matrix. It is denoted by $\mathcal{O}_{p\times q}$.

8. **Definition. (Additive inverse of a matrix.)**

   Let C be a $(p \times q)$-matrix with real entries.

   The $(p \times q)$-matrix $(-1)C$ is called the additive inverse of C. It is denoted by $-C$.

   Remark. In symbols, this definition says that if

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pq} \end{bmatrix}$$

then

$$-C = \begin{bmatrix} -c_{11} & -c_{12} & \cdots & -c_{1q} \\ -c_{21} & -c_{22} & \cdots & -c_{2q} \\ \vdots & \vdots & & \vdots \\ -c_{p1} & -c_{p2} & \cdots & -c_{pq} \end{bmatrix}.$$

9. **Definition. (Column vectors and row vectors.)**

   An $(p \times 1)$-matrix is called a column vector of size p.

   A $(1 \times q)$-matrix is called a row vector of size q.

   Remark. In this course, when we use the word *vector*, we mean *column vector* unless otherwise stated.

10. **Geometric visualization of vectors, and operations with vectors.**

    The key is the identification of vectors with their 'arrowheads'.

    We visualize the column vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ as an arrow with 'arrowhead' at the point $(c_1, c_2, \cdots, c_n)$ in real coordinate n-space and with tail at its origin.

    We may then further identify this vector as the point $(c_1, c_2, \cdots, c_n)$.

    - The column vector $[c_1]$ is identified as the point c_1 on the real line.
    - The column vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is identified as the point (c_1, c_2) on the real coordinate plane.
    - The column vector $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is identified as the point (c_1, c_2, c_3) on the real coordinate space. *Et cetera.*

    Remarks.

    (a) How to give a geometric interpretation of scalar multiplications for vectors?

(b) How to give a geometric interpretation of vector addition?

11. **Theorem (1). (Basic properties of matrix addition.)**

The statements below hold:

(a) *Suppose A, B, C are $(p \times q)$-matrices with real entries. Then $(A + B) + C = A + (B + C)$.*

(b) *Suppose A is a $(p \times q)$-matrix with real entries. Then $\mathcal{O}_{p\times q} + A = A = A + \mathcal{O}_{p\times q}$.*

(c) *Suppose A is a $(p \times q)$-matrix with real entries. Then $A + (-A) = \mathcal{O} = (-A) + A$.*

(d) *Suppose B, C are $(p \times q)$-matrices with real entries. Then $B + C = C + B$.*

Remark on terminologies.

(a) Statement (a) is known as the 'Law of Associativity' for matrix addition.

(b) Statement (b) is how the 'Law of Existence of Additive Identity' for matrix addition is manifested. The zero matrix is the 'additive identity' concerned.

(c) Statement (c) is how the 'Law of Existence of Additive inverse' for matrix addition is manifested. For each matrix A, the matrix $-A$ is its 'additive inverse'.

(d) Statement (d) is known as the 'Law of Commutativity' for matrix addition.

12. **Theorem (2). (Basic properties of matrix addition together with scalar multiplication.)**

The statements below hold:

(a) *Suppose C is a $(p \times q)$-matrix with real entries. Suppose α, β are real numbers. Then $\alpha(\beta C) = (\alpha\beta)C$.*

(b) *Suppose C is a $(p\times q)$-matrix with real entries. Suppose α, β are real numbers. Then $(\alpha+\beta)C = (\alpha C)+(\beta C)$.*

(c) *Suppose B, C are $(p\times q)$-matrices with real entries. Suppose α is a real number. Then $\alpha(B+C) = (\alpha B)+(\alpha C)$.*

(d) *Suppose C is a $(p \times q)$-matrix with real entries. Then $1C = C$.*

13. **Addition and scalar multiplication for 'block matrices', introduced through examples.**

(a) Let $A_1, \cdots, A_p$, $B_1, \cdots, B_p$ be matrices each with m rows. Suppose that for each k, the matrices A_k, B_k have the same number of columns.
Define $A = [\ A_1 \mid A_2 \mid \cdots \mid A_p\ ]$, and $B = [\ B_1 \mid B_2 \mid \cdots \mid B_p\ ]$.
Then $A + B = [\ A_1 + B_1 \mid A_2 + B_2 \mid \cdots \mid A_p + B_p\ ]$.
Moreover, for each $\alpha \in \mathbb{R}$, $\alpha A = [\ \alpha A_1 \mid \alpha A_2 \mid \cdots \mid \alpha A_p\ ]$.
Illustration.

Let $A = \left[\begin{array}{ccc|c|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{array}\right]$, $B = \left[\begin{array}{ccc|c|cc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \end{array}\right]$.

Let $A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}$, $A_3 = \begin{bmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{35} & a_{36} \\ a_{45} & a_{46} \end{bmatrix}$.

Let $B_1 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$, $B_2 = \begin{bmatrix} b_{14} \\ b_{24} \\ b_{34} \\ b_{44} \end{bmatrix}$, $B_3 = \begin{bmatrix} b_{15} & b_{16} \\ b_{25} & b_{26} \\ b_{35} & b_{36} \\ b_{45} & b_{46} \end{bmatrix}$.

Then we have $A = [\ A_1 \mid A_2 \mid A_3\ ]$, $B = [\ B_1 \mid B_2 \mid B_3\ ]$, and

$$A_1+B_1 = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \\ a_{41}+b_{41} & a_{42}+b_{42} & a_{43}+b_{43} \end{bmatrix}, A_2+B_2 = \begin{bmatrix} a_{14}+b_{14} \\ a_{24}+b_{24} \\ a_{34}+b_{34} \\ a_{44}+b_{44} \end{bmatrix}, A_3+B_3 = \begin{bmatrix} a_{15}+b_{15} & a_{16}+b_{16} \\ a_{25}+b_{25} & a_{26}+b_{26} \\ a_{35}+b_{35} & a_{36}+b_{36} \\ a_{45}+b_{45} & a_{46}+b_{46} \end{bmatrix}.$$

So $A + B = [\ A_1 + B_1 \mid A_2 + B_2 \mid A_3 + B_3\ ]$ indeed.

(b) Let $A_1, \cdots, A_p$, $B_1, \cdots, B_p$ be matrices each with n columns. Suppose that for each k, the matrices A_k, B_k have the same number of rows.

Define $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_p \end{array}\right]$, and $B = \left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_p \end{array}\right]$.

Then $A + B = \left[\begin{array}{c} A_1 + B_1 \\ \hline A_2 + B_1 \\ \hline \vdots \\ \hline A_p + B_p \end{array}\right]$.

Moreover, for each $\alpha \in \mathbb{R}$, $\alpha A = \begin{bmatrix} \alpha A_1 \\ \hline \alpha A_2 \\ \hline \vdots \\ \hline \alpha A_p \end{bmatrix}$.

(c) Let $A_{11}, A_{12}, A_{21}, A_{22}$, $B_{11}, B_{12}, B_{21}, B_{22}$ be matrices. Suppose that

- the number of rows of $A_{11}, A_{12}, B_{11}, B_{12}$ are the same,
- the number of rows of $A_{21}, A_{22}, B_{21}, B_{22}$ are the same,
- the number of columns of $A_{11}, A_{21}, B_{11}, B_{21}$ are the same, and
- the number of column of $A_{12}, A_{22}, B_{12}, B_{22}$ are the same.

Define $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$.

Then $A + B = \left[\begin{array}{c|c} A_{11} + B_{11} & A_{12} + B_{12} \\ \hline A_{21} + B_{21} & A_{22} + B_{22} \end{array}\right]$.

Moreover, for each $\alpha \in \mathbb{R}$, $\alpha A = \left[\begin{array}{c|c} \alpha A_{11} & \alpha A_{12} \\ \hline \alpha A_{21} & \alpha A_{22} \end{array}\right]$.

14. **Appendix 1. Proof of Theorem (1) and Theorem (2).**

Here we give the proof for one of the statements in Theorem (1) and Theorem (2).

The proofs for all other statements are similar.

Statement (c) of Theorem (2).

Suppose B, C are $(p \times q)$-matrices with real entries. Suppose α is a real number. Then $\alpha(B + C) = (\alpha B) + (\alpha C)$.

Proof of Statement (c) of Theorem (2).

Suppose B, C are $(p \times q)$-matrices with real entries.

For each i, j, denote the (i, j)-th entry of B by b_{ij}, and the (i, j)-th entry of B by c_{ij}.

Suppose α is a real number.

Fix any i, j.

- The (i, j)-th entry of $B + C$ is $b_{ij} + c_{ij}$.
  Then the (i, j)-th entry of $\alpha(B + C)$ is $\alpha(b_{ij} + c_{ij})$.
- The (i, j)-th entry of αB is αb_{ij}.
  The (i, j)-th entry of αC is αc_{ij}.
  Then the (i, j)-th entry of $(\alpha B) + (\alpha C)$ is $\alpha b_{ij} + \alpha c_{ij}$.
- Note that $\alpha(b_{ij} + c_{ij}) = \alpha b_{ij} + \alpha c_{ij}$ by the distributive laws for the reals.
  Then the (i, j)-th entry of $\alpha(B + C)$ and that of $(\alpha B) + (\alpha C)$ are the same.

It follows that $\alpha(B + C) = (\alpha B) + (\alpha C)$.

15. **Appendix 2. 'Algebraic laws of arithmetic' for the reals.** The 'algebraic laws of arithmetic' for addition and scalar multiplication, and those for matrix multiplication (to be introduced very soon) all rely on 'algebraic laws of arithmetic' (or the 'field laws') for the real number system:

(A1) *For any $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$.*

(A2) *For any $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$.*

(A3) *There exists some $z \in \mathbb{R}$, namely $z = 0$, such that for any $a \in \mathbb{R}$, $a + z = a$ and $z + a = a$.*

(A4) *For any $a \in \mathbb{R}$, there exists some $b \in \mathbb{R}$, called an* **additive inverse of** *a, such that $a + b = 0$ and $b + a = 0$.*

(A5) *For any $a, b \in \mathbb{R}$, $a + b = b + a$.*

(A6) *For any $a, b \in \mathbb{R}$, $a \times b \in \mathbb{R}$.*

(A7) *For any $a, b, c \in \mathbb{R}$, $(a \times b) \times c = a \times (b \times c)$.*

(A8) *There exists some $u \in \mathbb{R}$, namely $u = 1$, such that for any $a \in \mathbb{R}$, $a \times u = a$ and $u \times a = a$.*

(A9) *For any $a \in \mathbb{R}\backslash\{0\}$, there exists some $b \in \mathbb{R}$, called a* **multiplicative inverse of** *a, such that $a \times b = 1$ and $b \times a = 1$.*

(A10) *For any $a, b \in \mathbb{R}$, $a \times b = b \times a$.*

(A11) *For any $a, b, c \in \mathbb{R}$, $(a + b) \times c = (a \times c) + (b \times c)$ and $a \times (b + c) = (a \times b) + (a \times c)$.*

These 'laws' are also what make everything we have learnt about solving systems of linear equations work.