Further example (A)

For each real number α, consider the system of equations

$$(S_\alpha)\quad \begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 + 3x_2 = 4 \\ 3x_1 + \alpha x_2 = 5 \end{cases}.$$

We want to determine for which value(s) of α the system (S_α) is consistent, and for such a value of α, the solution set of (S_α).

Ask: Suppose (S_α) has a solution, say '$x_1 = a$, $x_2 = b$'.
What can we tell about a, b and α?
Is there some condition that a, b and α have to satisfy necessarily?

Answer. By the assumption here, we have these equalities about a, b and α:

$$\begin{cases} a + 2b = 3 & \text{--- ①} \\ 2a + 3b = 4 & \text{--- ②} \\ 3a + \alpha b = 5 & \text{--- ③} \end{cases}$$

In particular, ①, ② together give:

$$\begin{cases} a + 2b = 3 \\ 2a + 3b = 4. \end{cases}$$

Then $a = -1$ and $b = 2$.

Now ③ also gives:

$$5 = 3a + \alpha b = 3\cdot(-1) + \alpha \cdot 2 = 2\alpha - 3.$$

Then $\alpha = 4$.

So the necessary condition that a, b and α have to satisfy, under the assumption that (S_α) has '$x_1 = a$, $x_2 = b$' as a solution is:

$$a = -1 \text{ and } b = 2 \text{ and } \alpha = 4.$$

Hence we conclude:

If (S_α) has any solution at all, then $\alpha = 4$.

Further ask: Is it indeed true that if $\alpha = 4$ then (S_α) has a solution?

Or simply: Does (S_4) have any solution?

Answer. [This is your work.]

Solve (S_4). ...

(S_4) has a unique solution, namely $\begin{cases} x_1 = -1 \\ x_2 = 2 \end{cases}$.

Further example (B)

For each real number α, consider the system of equations

$$(S_\alpha)\quad \begin{cases} \quad\;\; x_2 + x_3 = 1 \\ x_1 \quad\;\; + x_3 = 0 \\ \alpha x_1 + x_2 \quad\;\; = 0 . \end{cases}$$

We want to determine for which value(s) of α the system (S_α) is consistent, and for such a value of α, the solution set of (S_α).

Ask: Suppose (S_α) has a solution, say '$x_1 = a$, $x_2 = b$, $x_3 = c$'.

What can we tell about a, b, c and α?

Is there some condition that a, b, c and α have to satisfy necessarily?

Answer. By the assumption here, we have these equalities about a, b, c and α:

$$\begin{cases} \quad\; b + c = 1 & \text{——— (1)} \\ a \quad\; + c = 0 & \text{——— (2)} \\ \alpha a + b \quad\; = 0 & \text{——— (3)} \end{cases}$$

In particular, (1), (2) together give:

$$\begin{cases} \quad b + c = 1 \\ a \quad + c = 0 . \end{cases}$$

Then $\begin{cases} a = -c \\ b = 1 - c . \end{cases}$

Now (3) also gives:

$$0 = \alpha a + b = \alpha(-c) + (1-c) = 1 - c(1+\alpha)$$

Therefore $c(1+\alpha) = 1$.

Then $c \neq 0$ and $\alpha \neq -1$.

Moreover, $c = \frac{1}{1+\alpha}$. Hence $a = -\frac{1}{1+\alpha}$ and $b = 1 - \frac{1}{1+\alpha}$.

So the necessary condition that a, b, c and α have to satisfy, under the assumption that (S_α) has '$x_1 = a$, $x_2 = b$, $x_3 = c$' as a solution is:

'$\alpha \neq -1$ and $c = \frac{1}{1+\alpha}$ and $a = -\frac{1}{1+\alpha}$ and $b = 1 - \frac{1}{1+\alpha}$.'

Hence we conclude:

If (S_α) has any solution at all, then $\alpha \neq -1$.

Further ask: Is it indeed true that if $\alpha \neq -1$ then (S_α) has a solution?

Answer. Suppose $\alpha \neq -1$. Solve (S_α). ... (S_α) has a unique solution, namely $\begin{cases} x_1 = -1/(1+\alpha) \\ x_2 = 1 - 1/(1+\alpha) \\ x_3 = 1/(1+\alpha) \end{cases}$.