1. **Definition. (Row-echelon form.)**

   Let C be a $(p \times q)$-matrix.

   C is said to be a row-echelon form if the statements below hold:

   (a) *All rows consisting of only zeros are at the bottom of C.*

   (b) *In every non-zero row, the first non-zero entry from the left is 1. (Such entries are called the leading ones of C.)*

   (c) *The leading one in each row is always strictly to the right of that in the row above it.*

   The number of leading ones in C is called the rank of the row-echelon form C.

   Remark. As a whole, the zeros in a row-echelon form to the left of the leading ones of the respective rows form something like a 'staircase' of zeros, and the leading ones a like step-edges of this 'staircase'. It reads like:

$$\left[\begin{array}{ccc|c|ccc|c|ccc|c|ccc|c|c}
0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star & \cdots & \star & \star & \star & \cdots & \star & \star & \star \\ \hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star & \cdots & \star & \star & \star \\ \hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star \\ \hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star \\ \hline
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \ddots \\ \hline
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right]$$

   The symbol $\mathbf{0}$ stands for a column of zeros.

2. **Examples of row-echelon forms.**

   These are row-echelon forms:

   (a) $\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\left[\begin{array}{cccccc} 1 & 3 & 5 & 7 & 9 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (c) $\left[\begin{array}{c|ccccc} 0 & 1 & 3 & 5 & 7 & 9 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

   Non-examples.

   These are not row-echelon forms:

   (a) $\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 0 \\ 0 & 0 & 4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 3 & 5 & 7 & 9 \\ 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. **Definition. (Reduced row-echelon form.)**

   Let C be a $(p \times q)$-matrix.

   C is said to be a reduced row-echelon form if C is a row-echelon form and furthermore:

   - *In any column which contains a leading one, the only non-zero entry is the entry provided by that leading one.*

   Such columns are called the pivot columns of C.

   The other columns of C are called the free columns of C.

   Remark. A reduced row-echelon form reads like:

$$\left[\begin{array}{ccc|c|ccc|c|ccc|c|ccc|c|c}
0 & \cdots & 0 & 1 & \star & \cdots & \star & 0 & \star & \cdots & \star & 0 & \star & \cdots & \star & 0 & \star \\ \hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & 0 & \star & \cdots & \star & 0 & \star \\ \hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & 0 & \star \\ \hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star \\ \hline
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \ddots \\ \hline
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right]$$

   Further convention and terminology. In such a reduced row-echelon form, say, C, the pivot columns whose leading ones are in the 1st, 2nd, 3rd, ... rows are respectively labelled as the d_1-th, d_2-th, d_3-th, ... columns. The free columns of C are labelled, from left to right, as the f_1-th, f_2-th, f_3-th, ... columns.

4. **Examples of reduced row-echelon forms.**

   These are reduced row-echelon forms:

(a) $\begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\left[\begin{array}{cccccc} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (c) $\left[\begin{array}{c|ccccc} 0 & 1 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Non-examples.

These are not reduced row-echelon forms, despite being row-echelon forms:

(a) $\begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\left[\begin{array}{cccccc} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (c) $\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Remark. Every reduced row-echelon form is a row-echelon form in the first place. If a matrix is not a row-echelon form, then it is definitely not a reduced row-echelon form.

5. **How to check whether a given matrix is a row-echelon form, or a reduced row-echelon form?**

Suppose C is a matrix.

(a) Inspect C and ask:

- Is it of the form $\left[\begin{array}{c|c} \mathcal{O} & \tilde{C} \\ \hline \mathcal{O} & \mathcal{O} \end{array}\right]$ for some matrix $\tilde{C}$ whose $(1,1)$-th entry is 1 and whose last row is non-zero?

(Here the $\mathcal{O}$'s stand for various rectangular arrays of 0's.)

(b) If no, C is not a row-echelon form.

If yes, inspect $\tilde{C}$ row by row, and ask:

- Is the leading one in each row always strictly to the right of that in the row above it?

(c) If no, C is not a row-echelon form.

If yes, C is a row echelon form (and $\tilde{C}$ is a row-echelon form). Now inspect the columns of $\tilde{C}$ which contain leading ones, and ask:

- Are they pivot columns of $\tilde{C}$?

(d) If no, C is not a reduced row-echelon form (and $\tilde{C}$ is a not a reduced row-echelon form).

If yes, C is a reduced row-echelon form (and $\tilde{C}$ is a reduced row-echelon form).

6. **Theorem (1). (Existence and uniqueness of reduced row-echelon form which is row-equivalent to a given matrix.)**

Suppose that A is a matrix (with real entries). Then the statements below hold:

(a) *There exists some row-echelon form B such that A is row-equivalent to B.*

(b) *There exists some unique reduced row-echelon form C such that A is row-equivalent to C.*

Proof. Omitted.

Remarks.

(1) The 'existence arguments' for both statements here can be given with the help of mathematical induction. They can be adapted to provide an 'algorithm', known as Gaussian elimination for matrices, which helps us obtain through the application of a sequence of row operations, from any given matrix, say, A, some row-echelon forms B, and furthermore a (unique) reduced row-echelon form C which are row-equivalent to the matrix A.

(2) The 'uniqueness argument' for reduced row-echelon form can also be given with the help of mathematical induction. This can be done efficiently with the help of the 'dictionary' between row operations and matrix multiplication by row-operation matrices, to be introduced later.

(3) Row-equivalence partitions the 'world' of all $(p \times q)$-matrices into 'chambers' of various $(p \times q)$-matrices.

Matrices within the same 'chambers' are row-equivalent to each other.

Two matrices of different chambers are not row-equivalent to each other.

The second part of Theorem (1) tells us that each such 'chamber' contains exactly one reduced row-echelon form which is row-equivalent to every other matrix in that 'chamber'.

7. **Gaussian elimination for matrices, introduced through an example.**

Consider the matrix

$$C = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}.$$

We are going to obtain, through a methodical application of row operations starting from C, first a row-echelon form and then a reduced row-echelon form which are row-equivalent to C.

(For a description of Gaussian elimination for matrices in the general situation, refer to the appendix.)

(a) We start by labelling C as C_1.

Unless C_1 is a row-echelon form, we apply appropriate row operations to obtain a matrix in which

- the left-most non-zero entry in the first row is 1.

$$C_1 = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C_2 = \begin{bmatrix} -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$$

$$\xrightarrow{-1R_5} C_3 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$$

(b) Consider the matrix C_3.

Unless C_3 is a row-echelon form, we apply appropriate row operations to obtain a matrix in which

- all non-zero entries below the first row are strictly to the right of the left-most entry in the first row.

$$C_3 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix} \xrightarrow{-2R_1+R_3} C_4 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$$

$$\xrightarrow{-3R_1+R_4} C_5 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 2 & 2 & 4 & -6 & 10 \end{bmatrix}$$

(c) Consider the matrix C_5. Leave the first row alone for now.

Unless C_5 is a row-echelon form, we apply appropriate row operations to obtain a matrix in which

- the left-most non-zero entry in the second row is 1, and
- all non-zero entries below the second row are strictly to the right of the left-most entry in the second row.

$$C_5 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 2 & 2 & 4 & -6 & 10 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C_6 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 0 & 0 & 2 & 2 & 4 & -6 & 10 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} C_7 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 2 & 4 & -6 & 10 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_4} C_8 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) Consider the matrix C_8. Leave the first two rows alone for now.

Unless C_8 is a row-echelon form, we apply appropriate row operations to obtain a matrix in which

- the left-most non-zero entry in the third row is 1, and
- all non-zero entries below the third row are strictly to the right of the left-most entry in the third row.

And so forth and so on.
We observe that C_8 is already a row-echelon form.

(e) Now we move on from C_8.

We apply appropriate row operations to 'eliminate' all the non-zero entries above the various leading ones in the non-zero rows of C_8.

$$C_8 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{1R_2+R_1} C_9 = \begin{bmatrix} 1 & -2 & 0 & 2 & 2 & -1 & 5 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_1} C_{10} = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-1R_3+R_2} C_{11} = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C_{11} is a reduced row-echelon form which is row-equivalent to C.

8. **Example (♯). (Gaussian elimination for matrices.)**

(a) (Compare this with what you see in Example (2) in the handout *What is solving a system of linear equations.*)

$$C_1 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-1R_1+R_2} C_2 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 6 & 5 & 6 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_3} C_3 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix} \xrightarrow{-2R_2+R_3} C_4 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

$$\xrightarrow{-1R_3} C_5 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-2R_2+R_1} C_6 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-1R_3+R_2} C_7 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

(b) (Compare this with what you see in Example (3) in the handout *What is solving a system of linear equations.*)

$$C_1 = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{bmatrix} \xrightarrow{-3R_1+R_2} C_2 = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ -1 & 3 & -5 & 3 \end{bmatrix}$$

$$\xrightarrow{1R_1+R_3} C_3 = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -4 & 5 \end{bmatrix} \xrightarrow{-2R_2+R_3} C_4 = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{1R_2+R_1} C_5 = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_3+R_1} C_6 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_3+R_2} C_7 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) (Compare this calculation with what you see in Example (4) in the handout *What is solving a system of linear equations.*)

$$C_1 = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C_2 = \begin{bmatrix} -1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 1 \\ 2 & 7 & -12 & 11 \end{bmatrix}$$

$$\xrightarrow{-1R_1} C_3 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 1 \\ 2 & 7 & -12 & 11 \end{bmatrix} \xrightarrow{-2R_1+R_3} C_4 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 3 & -6 & 3 \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_3} C_5 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2+R_1} C_6 = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) (Compare this calculation with what you see in Example (5) in the handout *What is solving a system of linear equations.*)

$$C_1 = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \xrightarrow{-1R_1+R_2} C_2 = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$$

$$\xrightarrow{-3R_1+R_3} C_3 = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{bmatrix} \xrightarrow{-1R_2} C_4 = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & -5 & 5 & -10 & -20 \end{bmatrix}$$

$$\xrightarrow{5R_2+R_3} C_5 = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_6+R_1} C_6 = \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) (Compare this calculation with what you see in Example (6) in the handout *What is solving a system of linear equations*.)

$$C_1 = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C_2 = \begin{bmatrix} 1 & 2 & 3 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{2R_1+R_3} C_3 = \begin{bmatrix} 1 & 2 & 3 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 3 & 7 & 7 & 11 \end{bmatrix} \xrightarrow{-3R_2+R_3} C_4 = \begin{bmatrix} 1 & 2 & 3 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_1} C_5 = \begin{bmatrix} 1 & 0 & 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{2R_3+R_1} C_6 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_2} C_7 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

Remark. When compared the respective calculations below to what you see in Examples (2), (3), (4), (5), (6) in the handout *What is solving a system of linear equations*, these calculations are seen to be applications of Gaussian elimination to the augmented matrix representations of the systems considered in the other handout.

In each calculation here, we reach the unique reduced row-echelon form row-equivalent to the matrix with which we start the calculation.

In the context of solving the systems in Examples (2), (4), (5), (6), the respective reduced row-echelon forms are significant in the sense that they correspond to systems from which we can read off solutions easily.

9. **Appendix: How Gaussian elimination is applied to a matrix to obtain a reduced row-echelon form.**

Suppose A is a matrix. We proceed to apply row operations to obtain successive matrices which are row-equivalent to A, until we obtain a reduced row-echelon form which is row equivalent to A.

(a) If A is a row-echelon form, take $\hat{A} = A$ and go straight to **Procedure** ω.

(b) Suppose A is not a row-echelon form. (Then A contains some non-zero entry.) Proceed to obtain a matrix, labelled $\hat{A}$, which is row-equivalent to A as follows:—

i. **Procedure** α.

A. If A reads as

$$A = \left[\begin{array}{ccc|c|c} 0 & \cdots & 0 & \dagger & \star \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \star & \star \end{array}\right]$$

in which $\dagger$ is a non-zero number, then take $A' = A$.
If not, obtain from A, by applying the row operation '$R_1 \leftrightarrow R_k$' for some appropriate k, a matrix which reads so. Label the resultant matrix as A'.

B. If A' reads as

$$A' = \left[\begin{array}{ccc|c|c} 0 & \cdots & 0 & 1 & \star \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \star & \star \end{array}\right],$$

then take $A'' = A'$.
If not, obtain from A', by applying the row operation 'βR_1' for some appropriate β, the matrix which reads so. Label the result matrix as A''.

C. If A'' reads as

$$A_* = \left[\begin{array}{ccc|c|c} 0 & \cdots & 0 & 1 & \star \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & B \end{array}\right]$$

then take $A_* = A''$.
If not, obtain from A'', by (repeatedly) applying row operations of the type '$\alpha R_1 + R_k$' to obtain the matrix which reads so. Label the resultant matrix as A_*.

With the matrix A_* go into **Procedure** β.

ii. **Procedure** β.

Consider the matrix B 'sitting inside A_*'.

If B is a row-echelon form, then A_* is a row echelon form. We take $A_* = \hat{A}$ and go straight to **Procedure** ω.
Suppose B is not a row-echelon form. Then obtain a matrix, labelled A_{**}, from A_* by this 'sub-procedure':

A. Replace the symbol 'A' with the symbol 'B' in **Procedure** α to obtain from B a matrix B_* which reads:

$$B_* = \left[\begin{array}{ccc|c|c} 0 & \cdots & 0 & 1 & \star \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & C \end{array}\right].$$

B. Now obtain the matrix A_{**}, with the same number of rows and the same number of columns as A_*, which reads:

$$A_{**} = \left[\begin{array}{ccc|c|c} 0 & \cdots & 0 & 1 & \star \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & B_* \end{array}\right] = \left[\begin{array}{ccc|c|ccc|c|c} 0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & C \end{array}\right].$$

iii. If C is a row-echelon form, then A_{**} is a row-echelon form. We take $A_{**} = \hat{A}$ and then go straight to **Procedure** ω.
If C is not a row-echelon form, then imitate the 'sub-procedure' inside **Procedure** β to obtain successive matrices $A_{***}, A_{****}, \ldots$, with the same number of rows and the same number of columns, which are row-equivalent to A, until a row-echelon form is obtained:

$$A_{***} = \left[\begin{array}{ccc|c|ccc|c|ccc|c|c} 0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star & \cdots & \star & \star & \star \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & D \end{array}\right],$$

$$A_{****} = \left[\begin{array}{ccc|c|ccc|c|ccc|c|ccc|c|c} 0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star & \cdots & \star & \star & \star & \cdots & \star & \star & \star \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star & \cdots & \star & \star & \star \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \star & \star \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \star \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & E \end{array}\right]$$

$$\vdots$$

Label, as $\hat{A}$, the resultant row-echelon form, which is by construction row equivalent to A. Then go to **Procedure** ω.

(c) **Procedure** ω.

Consider the row-echelon form $\hat{A}$ which is row-equivalent to A.

i. If $\hat{A}$ is a reduced row-echelon form, then 'stop'.

ii. Suppose $\hat{A}$ is not a reduced row-echelon form.
For each row which contains a leading one, apply row operations of the type '$\alpha R_k + R_i$' repeatedly to obtain a reduced row-echelon form which is row-equivalent to $\hat{A}$ and to A.