1. **Definition. (Matrix.)**

A $(p \times q)$*-rectangular array*

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pq} \end{bmatrix}$$

in which the c_{ij}*'s are real numbers is called a* $(p \times q)$*-matrix with real entries, with* p *rows and* q *columns.*

Suppose We denote this matrix by C*.*

For each fixed $i = 1, 2, \cdots, p$ *and* $j = 1, 2, \cdots, q$*, we refer to the number* c_{ij} *as the* (i, j)*-th entry of* C*.*

For each fixed $k = 1, 2, \cdots, p$*, we refer to the array* $\begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kq} \end{bmatrix}$*, as the* k*-th row of* C*.*

For each fixed $\ell = 1, 2, \cdots, q$*, we refer to the array* $\begin{bmatrix} c_{1\ell} \\ c_{2\ell} \\ \vdots \\ c_{q\ell} \end{bmatrix}$*, as the* ℓ*-th column of* C*.*

2. **Definition. (Row operation 'adding a scalar multiple of one row to another'.)**

Let C be a $(p \times q)$-matrix whose (i, j)-th entry is denoted by c_{ij}, and whose k-th row is denoted by R_k.

Suppose α is a real number.

When we replace the k-th row $\begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kq} \end{bmatrix}$ of C by

$$\begin{bmatrix} \alpha c_{i1} + c_{k1} & \alpha c_{i2} + c_{k2} & \cdots & \alpha c_{iq} + c_{kq} \end{bmatrix},$$

in which $i \neq k$, to obtain the resultant matrix C', we say we are applying the row operation '$\alpha \cdot R_i + R_k$' to C, and write $C \xrightarrow{\alpha R_i + R_k} C'$.

Such a row operation is called 'adding a scalar multiple of one row of C to another row of C'.

3. **Examples on 'adding a scalar multiple of one row to another'.**

(a) $C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1 + R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

(b) $C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2 + R_1} C'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

2. **Definition. (Row operation 'adding a scalar multiple of one row to another'.)**

Let C be a $(p \times q)$-matrix whose (i,j)-th entry is denoted by c_{ij}, and whose k-th row is denoted by R_k.

Suppose α is a real number.

When we replace the k-th row $\begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kq} \end{bmatrix}$ of C by

$$\begin{bmatrix} \alpha c_{i1} + c_{k1} & \alpha c_{i2} + c_{k2} & \cdots & \alpha c_{iq} + c_{kq} \end{bmatrix},$$

in which $i \neq k$, to obtain the resultant matrix C', we say we are applying the row operation '$\alpha \cdot R_i + R_k$' to C, and write $C \xrightarrow{\alpha R_i + R_k} C'$.

Such a row operation is called 'adding a scalar multiple of one row of C to another row of C'.

3. **Examples on 'adding a scalar multiple of one row to another'.**

(a) $C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$

(b) $C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2+R_1} C'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$

4. **Definition. (Row operation 'multiplying a non-zero scalar to a row'.)**

Let C be a $(p \times q)$-matrix whose (i, j)-th entry is denoted by c_{ij}, and whose k-th row is denoted by R_k.

Suppose β is a non-zero real number.

When replace the k-th row $\begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kq} \end{bmatrix}$ of C by

$$\begin{bmatrix} \beta c_{k1} & \beta c_{k2} & \cdots & \beta c_{kq} \end{bmatrix}$$

to obtain the resultant matrix C', we say we are applying the row operation '$\beta \cdot R_k$' to C, and write $C \xrightarrow{\beta R_k} C'$.

Such a row operation is called 'multiplying a non-zero scalar to a row of C'.

5. **Examples on 'multiplying a non-zero scalar to a row'.**

(a) $C = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

(b) $C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} C'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

4. **Definition. (Row operation 'multiplying a non-zero scalar to a row'.)**

Let C be a $(p \times q)$-matrix whose (i, j)-th entry is denoted by c_{ij}, and whose k-th row is denoted by R_k.

Suppose β is a non-zero real number.

When replace the k-th row $\begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kq} \end{bmatrix}$ of C by

$$\begin{bmatrix} \beta c_{k1} & \beta c_{k2} & \cdots & \beta c_{kq} \end{bmatrix}$$

to obtain the resultant matrix C', we say we are applying the row operation '$\beta \cdot R_k$' to C, and write $C \xrightarrow{\beta R_k} C'$.

Such a row operation is called 'multiplying a non-zero scalar to a row of C'.

5. **Examples on 'multiplying a non-zero scalar to a row'.**

(a) $C = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

(b) $C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} C'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

6. **Definition. (Row operation 'interchanging two rows'.)**

Let C be a $(p \times q)$-matrix whose (i, j)-th entry is denoted by c_{ij}, and whose k-th row is denoted by R_k.

When we interchange the i-th row $\begin{bmatrix} c_{i1} & c_{i2} & \cdots & c_{iq} \end{bmatrix}$ and the k-th row $\begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kq} \end{bmatrix}$ of C, in which $i \neq k$, to obtain the resultant matrix C', we say we are applying the row operation '$R_i \longleftrightarrow R_k$' to C, and write $C \xrightarrow{R_i \leftrightarrow R_k} C'$.

Such a row operation is called 'interchanging two rows of C'.

7. **Examples on 'interchanging two rows'.**

(a) $C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$.

(b) $C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}$.

6. **Definition. (Row operation 'interchanging two rows'.)**

Let C be a $(p \times q)$-matrix whose (i, j)-th entry is denoted by c_{ij}, and whose k-th row is denoted by R_k.

When we interchange the i-th row $\begin{bmatrix} c_{i1} & c_{i2} & \cdots & c_{iq} \end{bmatrix}$ and the k-th row $\begin{bmatrix} c_{k1} & c_{k2} & \cdots & c_{kq} \end{bmatrix}$ of C, in which $i \neq k$, to obtain the resultant matrix C', we say we are applying the row operation '$R_i \longleftrightarrow R_k$' to C, and write $C \xrightarrow{R_i \leftrightarrow R_k} C'$.

Such a row operation is called 'interchanging two rows of C'.

7. **Examples on 'interchanging two rows'.**

(a) $C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$

(b) $C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}.$

8. **Definition. (Row operations.)**

Let C, C' be $(p \times q)$-matrices with real entries.

We say we are applying one row operation on C to obtain C' if and only if C' is the resultant of the application of

- *one row operation 'adding a scalar multiple of one row of C to another row of C', or*
- *one row operation 'multiplying a non-zero scalar to a row of C', or*
- *one row operation 'interchanging two rows of C'.*

9. **Definition. (Sequences of row operations.)**

Let $C_1, C_2, \cdots, C_{N-1}, C_N$ *be finitely many* $(p \times q)$*-matrices.*

Suppose that for each k*,* C_{k+1} *is the resultant of the application of one row operation on* C_k*.*

Then we say that $C_1, C_2, \cdots, C_{N-1}, C_N$ *is joint by a sequence of row operations.*

When we want to emphasize that for each k*, the row operation* ρ_k *is applied to* C_k *to obtain* C_{k+1}*, we will present this sequence as*

$$C_1 \underset{\rho_1}{\longrightarrow} C_2 \underset{\rho_2}{\longrightarrow} \cdots \underset{\rho_{N-2}}{\longrightarrow} C_{N-1} \underset{\rho_{N-1}}{\longrightarrow} C_N.$$

We may also refer to such a sequence as the sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{N-1}$ *when we want to emphasize the role of the row operations.*

Remark. When $N = 1$, we have the 'trivial sequence' of row operations 'C_1'.

10. **Examples on sequences of row operations.**

(a) $$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2+R_1} C'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(b) $$C = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} C'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(c) $$C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}.$$

(d) $$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_3} C'' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix}$$
$$\xrightarrow{R_1 \leftrightarrow R_3} C''' = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

11. **Theorem (1). (Existence and uniqueness of 'reverse row operations'.)**
Let C, C' be $(p \times q)$-matrices.

Suppose C' is obtained from C by the application of a row operation ρ on C.

Then there exists some unique row operation $\tilde{\rho}$ on C' such that C is obtained from C' by the application of $\tilde{\rho}$ on C'.

Proof. A tedious (but easy) word game playing with the definitions.

Remark. We refer to $\tilde{\rho}$ as the 'reverse' row operation corresponding to ρ.

Row operation changing C to C'.	'Reverse' row operation changing C' to C.
$C \xrightarrow{\alpha R_i + R_k} C'$.	$C' \xrightarrow{-\alpha R_i + R_k} C$.
$C \xrightarrow{\beta R_k} C'$.	$C' \xrightarrow{(1/\beta) R_k} C$.
$C \xrightarrow{R_i \leftrightarrow R_k} C'$.	$C' \xrightarrow{R_i \leftrightarrow R_k} C$.

11. **Theorem (1). (Existence and uniqueness of 'reverse row operations'.)**
Let C, C' be $(p \times q)$-matrices.

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Proof. A tedious (but easy) word game playing with the definitions.

Remark. We refer to $\tilde{\rho}$ as the 'reverse' row operation corresponding to ρ.

Illustrations.

Row operation changing C to C'.	'Reverse' row operation changing C' to C.
$C \xrightarrow{\alpha R_i + R_k} C'$.	$C' \xrightarrow{-\alpha R_i + R_k} C$.
$C \xrightarrow{\beta R_k} C'$.	$C' \xrightarrow{(1/\beta) R_k} C$.
$C \xrightarrow{R_i \leftrightarrow R_k} C'$.	$C' \xrightarrow{R_i \leftrightarrow R_k} C$.

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1 + R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1R_1 + R_2} C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{4}R_2} C = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

12. **Examples.**

(a) $$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2+R_1} C'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Corresponding reverse row operations, 'recovering' C from C'':

$$C'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_2+R_1} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1R_1+R_2} C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(b) $$C = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} C'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Corresponding reverse row operations, 'recovering' C from C'':

$$C'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} C' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2} C = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(c) $C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}.$

Corresponding reverse row operations, 'recovering' C from C'':

$$C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$$

(d) $C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_3} C'' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix}$

$$\xrightarrow{R_1 \leftrightarrow R_3} C''' = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Corresponding reverse row operations, 'recovering' C from C''':

$$C''' = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} C'' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{(1/2)R_3} C' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{-1R_1+R_2} C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

These examples are manifestations of the phenomenon described by Theorem (2).

13. **Theorem (2).**

Suppose $C_1, C_2, \cdots, C_N$ *is a sequence of* $(p \times q)$*-matrices joint by row operations*

$$\rho_1, \rho_2, \cdots, \rho_{N-1}$$

respectively:

$$C_1 \underset{\rho_1}{\longrightarrow} C_2 \underset{\rho_2}{\longrightarrow} \cdots \underset{\rho_{N-2}}{\longrightarrow} C_{N-1} \underset{\rho_{N-1}}{\longrightarrow} C_N.$$

Then $C_N, \cdots, C_2, C_1$ *is a sequence of* $(p \times q)$*-matrices joint by row operations*

$$\widetilde{\rho_{N-1}}, \cdots, \widetilde{\rho_2}, \widetilde{\rho_1}$$

respectively, in which $\widetilde{\rho_k}$ *is the 'reverse row operation' of* ρ_k *for each* k*:*

$$C_N \underset{\widetilde{\rho_{N-1}}}{\longrightarrow} C_{N-1} \underset{\widetilde{\rho_{N-2}}}{\longrightarrow} \cdots \underset{\widetilde{\rho_2}}{\longrightarrow} C_2 \underset{\widetilde{\rho_1}}{\longrightarrow} C_1.$$

Proof. The argument is a repeated application of Theorem (1).

14. **Definition. (Row-equivalent matrices.)**

Let C, D *be* $(p \times q)$*-matrices.*

Suppose there is a finite sequence of row operations starting from C *and ending at* D*.*

Then we say that C *is row-equivalent to* D*.*

15. **Question.** How to show that a given $(p\times q)$-matrix C is row-equivalent to a $(p\times q)$-matrix D?

Answer. Write down a finite sequence of $(p \times q)$-matrices, say,

$$C = C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_{N-1} \longrightarrow C_N = D$$

joint by row operations, one at each step.

Illustration.

Let $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ *and* $D = \begin{bmatrix} 3 & 6 & 6 \\ 8 & -8 & 4 \end{bmatrix}$.

We verify that C is row-equivalent to D:

$$\begin{aligned} C = C_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad & \xrightarrow{1R_1+R_2} C_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \\ & \xrightarrow{3R_1} C_3 = \begin{bmatrix} 3 & 0 & 3 \\ 1 & 2 & 2 \end{bmatrix} \\ & \xrightarrow{R_1 \leftrightarrow R_2} C_4 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 3 \end{bmatrix} \\ & \xrightarrow{-1R_1+R_2} C_5 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix} \\ & \xrightarrow{4R_2} C_6 = \begin{bmatrix} 1 & 2 & 2 \\ 8 & -8 & 4 \end{bmatrix} \\ & \xrightarrow{3R_1} C_7 = D = \begin{bmatrix} 3 & 6 & 6 \\ 8 & -8 & 4 \end{bmatrix} \end{aligned}$$

16. **Theorem (3). (Row-equivalence as an 'equivalence relation'.)**

The statements below hold:

(a) *Suppose C is a $(p \times q)$-matrix. Then C is row-equivalent to C.*

(b) *Let C, D be $(p \times q)$-matrices. Suppose C is row-equivalent to D. Then D is row-equivalent to C.*

(c) *Let C, D, E be $(p \times q)$-matrices. Suppose C is row-equivalent to D and D is row-equivalent to E. Then C is row-equivalent to E.*

Proof. Exercise.

Remark. According to Theorem (3), the collection of all $(p \times q)$-matrices are split into various 'cliques' according to the question whether one $(p \times q)$-matrix is row-equivalent to another $(p \times q)$-matrix. If yes, then the two matrices concerned are in the same 'clique'; if no, they are not.

17. **Appendix: column operations.**

There are 'mathematical objects' known as column operations on matrices. They are defined in an analogous way as row operations. There are analogous results on column operations. In this course most of the time we will need row operations only. There is no need to worry about column operations until further noticed.