MATH1030 What is solving a system of linear equations?

0. We take for granted that you have a rough idea from school days what you have to in order to solve a given ‘system of two linear equations with two unknowns’ (or what you call a ‘system of simultaneous linear equations’).

   Here we are going to explain, through a concrete example, the mathematical reasons behind the method you apply, and the format of presentation of answer.

   The same reasoning applies when we extend the method for solving systems of two linear equations with two unknowns to solving systems of many equations with many unknowns. We shall see this through several concrete examples of systems of three linear equations with three unknowns.

1. **Example (1). (On ‘solving’ a system of linear equations, from school mathematics.)**

   (a) Consider this *question from school maths*:

   What are the solutions of the system of equations $\left\{\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 3 \\ 2x_1 & - & x_2 & = & 4 \end{array}\right.$ *with unknowns* x_1, x_2 *in the reals?*

   Below is a *typical answer to this question in a school maths textbook.* (Such an answer likely omits the words put inside the ‘square brackets’.)

   [We search for candidate solution for the system concerned, labelled by (S_1) here:]

$$\begin{array}{rcl} & (S_1) & \left\{\begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & = & 3 & — & \textcircled{1} \\ x_1 & - & x_2 & = & 4 & — & \textcircled{2} \end{array}\right. \\ (-2)\times\textcircled{2}+\textcircled{1}: & (S_2) & \left\{\begin{array}{rcrcrcl} & & 5x_2 & = & -5 & — & \textcircled{3} \\ x_1 & - & x_2 & = & 4 & — & \textcircled{2} \end{array}\right. \\ \frac{1}{5}\times\textcircled{3}: & (S_3) & \left\{\begin{array}{rcrcrcl} & & x_2 & = & -1 & — & \textcircled{4} \\ x_1 & - & x_2 & = & 4 & — & \textcircled{2} \end{array}\right. \\ 1\times\textcircled{4}+\textcircled{2}: & (S_4) & \left\{\begin{array}{rcrcr} & & x_2 & = & -1 \\ x_1 & & & = & 3 \end{array}\right. \end{array}$$

   [We check the candidate solution is indeed a solution:
   Substituting $(x_1, x_2) = (3, -1)$ into the equations $\textcircled{1}$, $\textcircled{2}$ in (S_1), we obtain valid equalities.]
   The solution of (S_1) is given by $(x_1, x_2) = (3, -1)$.

   (b) **Question.** What have we done actually in ‘answering’ the question concerned in the format above?

   Answer. What we have done is ‘scrambling’ the three-step process of calculations and reasoning described below:

   - *Step 1.* Search for a candidate solution for the system of equations (S_1), through asking the question (†):

     (†) *Suppose* x_1, x_2 *are concrete real numbers for which the equalities* $2x_1 + 3x_2 = 3$ *and* $x_1 - x_2 = 4$ *hold simultaneously. Which numbers can* x_1, x_2 *be?*

     At the end of Step 1, we have the answer for (†):

     It is possible for (x_1, x_2) *to be* $(3, -1)$, *and there is no other possibility.*

   - *Step 2.* Check whether the ‘candidate solution’ obtained at the end of Step 1 is indeed a solution for the problem, through asking the question (‡):

     (†) *Suppose* $x_1 = 3$ *and* $x_2 = -1$. *Do the equalities* $2x_1 + 3x_2 = 3$ *and* $x_1 - x_2 = 4$ *hold simultaneously?*

     At the end of Step 2, we have the answer for (‡), which is *yes*.

   - *Step 3.* Draw the conclusion: *The solution of the problem is given by* $(x_1, x_2) = (3, -1)$.

   (c) **Detail of this the three-step process of calculations and reasoning?**

   Step 1. Searching for ‘candidate solution’ for the system of equations.

   [Ask. Suppose x_1, x_2 are concrete real numbers for which the equalities $2x_1 + 3x_2 = 3$ and $x_1 - x_2 = 4$ hold simultaneously. Which numbers can x_1, x_2 be?]

$$\begin{array}{lrcrlrcl} \text{Suppose} & 2x_1 + 3x_2 & = & 3 & \text{and} & x_1 - x_2 & = & 4. \\ \text{Then} & 5x_2 & = & -5 & \text{and} & x_1 - x_2 & = & 4. \\ \text{Then} & x_2 & = & -1 & \text{and} & x_1 - x_2 & = & 4. \\ \text{Then} & x_2 & = & -1 & \text{and} & x_1 & = & 3. \end{array}$$

   Step 2. Checking ‘candidate solution’.

   [Ask. Suppose $x_1 = 3$ and $x_2 = -1$. Is it true that $2x_1 + 3x_2 = 3$ and $x_1 - x_2 = 4$ simultaneously?]
   When $x_1 = 3$ and $x_2 = -1$, we have $2x_1 + 3x_2 = 2 \cdot 3 + 3 \cdot (-1) = 3$ and $x_1 - x_2 = 3 - (-1) = 4$.

   Step 3. Drawing conclusion.

   The solution of the system of equations $\left\{\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 3 \\ 2x_1 & - & x_2 & = & 4 \end{array}\right.$ is given by $(x_1, x_2) = (3, -1)$.

2. **Example (2).**

What are the solutions of the system of equations $\left\{\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array}\right.$ *with unknowns* x_1, x_2, x_3 *in the reals, if there is any at all?*

(a) The answer to this question is the three-step process below:

Step 1. Searching for 'candidate solution' for the system of equations.

[Ask. Suppose x_1, x_2, x_3 are concrete real numbers for which the equalities

$$\left\{\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array}\right.$$

hold simultaneously. Which numbers can x_1, x_2, x_3 be?]

For such numbers x_1, x_2, x_3, we obtain the equalities below, in succession:

$$(S_1) \quad \left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & — & \textcircled{1} \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 & — & \textcircled{2} \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 & — & \textcircled{3} \end{array}\right.$$

Then by $(-1) \times \textcircled{1} + \textcircled{2}$ we obtain :
$$(S_2) \quad \left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & — & \textcircled{1} \\ & & x_2 & + & x_3 & = & 1 & — & \textcircled{4} \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 & — & \textcircled{3} \end{array}\right.$$

Then by $(-2) \times \textcircled{1} + \textcircled{3}$ we obtain :
$$(S_3) \quad \left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & — & \textcircled{1} \\ & & x_2 & + & x_3 & = & 1 & — & \textcircled{4} \\ & & 2x_2 & + & x_3 & = & -2 & — & \textcircled{5} \end{array}\right.$$

Then by $(-2) \times \textcircled{4} + \textcircled{5}$ we obtain :
$$(S_4) \quad \left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & — & \textcircled{1} \\ & & x_2 & + & x_3 & = & 1 & — & \textcircled{4} \\ & & & - & x_3 & = & -4 & — & \textcircled{6} \end{array}\right.$$

Then by $(-1) \times \textcircled{6}$ we obtain :
$$(S_5) \quad \left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & — & \textcircled{1} \\ & & x_2 & + & x_3 & = & 1 & — & \textcircled{4} \\ & & & & x_3 & = & 4 & — & \textcircled{7} \end{array}\right.$$

Then by $(-2) \times \textcircled{4} + \textcircled{1}$ we obtain :
$$(S_6) \quad \left\{\begin{array}{rcrcrcrcl} x_1 & & & & & = & 2 & — & \textcircled{8} \\ & & x_2 & + & x_3 & = & 1 & — & \textcircled{4} \\ & & & & x_3 & = & 4 & — & \textcircled{7} \end{array}\right.$$

Then by $(-1) \times \textcircled{7} + \textcircled{4}$ we obtain :
$$(S_7) \quad \left\{\begin{array}{rcrcrcrcl} x_1 & & & & & = & 2 & — & \textcircled{8} \\ & & x_2 & & & = & -3 & — & \textcircled{9} \\ & & & & x_3 & = & 4 & — & \textcircled{7} \end{array}\right.$$

Hence we conclude: *it is possible for* (x_1, x_2, x_3) *to be* $(2, -3, 4)$ *and there is no other possibility.*

Step 2. Checking 'candidate solution'.

[Ask. Suppose $x_1 = 2$ and $x_2 = -3$ and $x_3 = 4$. Is it true that $x_1 + 2x_2 + 2x_3 = 4$ and $x_1 + 3x_2 + 3x_3 = 5$ and $2x_1 + 6x_2 + 5x_3 = 6$ simultaneously?]

When $x_1 = 2$ and $x_2 = -3$ and $x_3 = 4$, we have:

$x_1 + 2x_2 + 2x_3 = 2 + 2(-3) + 2 \cdot 4 = 4$, and

$x_1 + 3x_2 + 3x_3 = 2 + 3(-3) + 3 \cdot 4 = 5$, and

$2x_1 + 6x_2 + 5x_3 = 2 \cdot 2 + 6(-3) + 5 \cdot 4 = 6$.

Step 3. Drawing conclusion.

The solution of the system of equations $\left\{\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array}\right.$ is given by $(x_1, x_2, x_3) = (2, -3, 4)$.

(b) For this problem, it will transpire (after we learn a bit more theory) that when we are only required to determine the solution of the system, then (with the theoretical results already covering the reasoning in this three-step process,) it will suffice to present as below:

Label the system to be solved by (S_1).

$$(S_1) \quad \left\{ \begin{array}{rcrcrcrcc} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & — & ① \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 & — & ② \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 & — & ③ \end{array} \right.$$

$$(-1) \times ① + ② : \qquad (S_2) \quad \left\{ \begin{array}{rcrcrcrcc} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & — & ① \\ & & x_2 & + & x_3 & = & 1 & — & ④ \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 & — & ③ \end{array} \right.$$

$$\vdots \qquad \vdots$$

$$(-1) \times ⑦ + ④ : \qquad (S_7) \quad \left\{ \begin{array}{lllcrcc} x_1 & & & = & 2 & — & ⑧ \\ & x_2 & & = & -3 & — & ④ \\ & & x_3 & = & 4 & — & ⑦ \end{array} \right.$$

The solution of the system (S_1) *is given by* $(x_1, x_2, x_3) = (2, -3, 4)$.

3. **Example (3).**

What are the solutions of the system of equations $\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 7 \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 \end{array}\right.$ *with unknowns* x_1, x_2, x_3 *in the reals, if there is any at all?*

(a) The answer to this question is the three-step process below:

Step 1. Searching for 'candidate solution' for the system of equations.

[Ask. Suppose x_1, x_2, x_3 are concrete real numbers for which the equalities

$$\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 7 \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 \end{array}\right.$$

hold simultaneously. Which numbers can x_1, x_2, x_3 be?]

For such numbers x_1, x_2, x_3, we obtain the equalities below, in succession:

$$(S_1) \quad \left\{\begin{array}{rcrcrcrcc} x_1 & - & x_2 & + & x_3 & = & 2 & — & ① \\ 3x_1 & - & 2x_2 & + & x_3 & = & 7 & — & ② \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 & — & ③ \end{array}\right.$$

$$\text{Then by } (-3) \times ① + ② \text{ we obtain :} \quad (S_2) \quad \left\{\begin{array}{rcrcrcrcc} x_1 & - & x_2 & + & x_3 & = & 2 & — & ① \\ & & x_2 & - & 2x_3 & = & 1 & — & ④ \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 & — & ③ \end{array}\right.$$

$$\text{Then by } 1 \times ① + ③ \text{ we obtain :} \quad (S_3) \quad \left\{\begin{array}{rcrcrcrcc} x_1 & - & x_2 & + & x_3 & = & 2 & — & ① \\ & & x_2 & - & 2x_3 & = & 1 & — & ④ \\ & & 2x_2 & - & 4x_3 & = & 5 & — & ⑤ \end{array}\right.$$

$$\text{Then by } (-2) \times ④ + ⑤ \text{ we obtain :} \quad (S_4) \quad \left\{\begin{array}{rcrcrcrcc} x_1 & - & x_2 & + & x_3 & = & 2 & — & ① \\ & & x_2 & - & 2x_3 & = & 1 & — & ④ \\ & & & & 0 & = & 1 & — & ⑥ \end{array}\right.$$

This tells us that it would happen (under the assumption of the existence of such real numbers x_1, x_2, x_3) that $0 = 1$.

But we know '$0 = 1$' cannot happen in the first place.

Hence we conclude: *there is no 'candidate solution' for the system.*

Step 2. Checking 'candidate solution'.

There is no need of such a step, as there is no 'candidate solution' to talk about.

Step 3. Drawing conclusion.

The system of equations $\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 7 \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 \end{array}\right.$ has no solution.

(b) For this problem, it will transpire (after we learn a bit more theory) that when we are only required to determine whether there is any solution of the system, then (with the theoretical results already covering the reasoning in this three-step process,) it will suffice to present as below:

Label the system to be solved by (S_1).

$$(S_1) \quad \left\{\begin{array}{rcrcrcrcc} x_1 & - & x_2 & + & x_3 & = & 2 & — & ① \\ 3x_1 & - & 2x_2 & + & x_3 & = & 7 & — & ② \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 & — & ③ \end{array}\right.$$

$$(-3) \times ① + ② : \quad (S_2) \quad \left\{\begin{array}{rcrcrcrcc} x_1 & - & x_2 & + & x_3 & = & 2 & — & ① \\ & & x_2 & - & 2x_3 & = & 1 & — & ④ \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 & — & ③ \end{array}\right.$$

$$\vdots \quad \vdots$$

$$(-2) \times ④ + ⑤ : \quad (S_4) \quad \left\{\begin{array}{rcrcrcrcc} x_1 & - & x_2 & + & x_3 & = & 2 & — & ① \\ & & x_2 & - & 2x_3 & = & 1 & — & ④ \\ & & & & 0 & = & 1 & — & ⑥ \end{array}\right.$$

The system (S_1) *has no solution.*

4. **Example (4).**

What are the solutions of the system of equations $\left\{\begin{array}{rcrcrcl} & & x_2 & - & 2x_3 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 \end{array}\right.$ *with unknowns* x_1, x_2, x_3 *in the reals, if there is any at all?*

(a) The answer to this question is the three-step process below:

Step 1. Searching for 'candidate solution' for the system of equations.

[Ask. Suppose x_1, x_2, x_3 are concrete real numbers for which the equalities

$$\left\{\begin{array}{rcrcrcl} & & x_2 & - & 2x_3 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 \end{array}\right.$$

hold simultaneously. Which numbers can x_1, x_2, x_3 be?]

For such numbers x_1, x_2, x_3, we obtain the equalities below, in succession:

$$(S_1) \quad \left\{\begin{array}{rcrcrcrl} & & x_2 & - & 2x_3 & = & 1 & — ① \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 & — ② \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 & — ③ \end{array}\right.$$

Then by ① ↔ ② we obtain : $(S_2) \quad \left\{\begin{array}{rcrcrcrl} -x_1 & - & 2x_2 & + & 3x_3 & = & -4 & — ④ \\ & & x_2 & - & 2x_3 & = & 1 & — ⑤ \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 & — ③ \end{array}\right.$

Then by $(-1) \times$ ④ we obtain : $(S_3) \quad \left\{\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & - & 3x_3 & = & 4 & — ⑥ \\ & & x_2 & - & 2x_3 & = & 1 & — ⑤ \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 & — ③ \end{array}\right.$

Then by $(-2) \times$ ⑥ $+$ ③ we obtain : $(S_4) \quad \left\{\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & - & 3x_3 & = & 4 & — ⑥ \\ & & x_2 & - & 2x_3 & = & 1 & — ⑤ \\ & & 3x_2 & - & 6x_3 & = & 3 & — ⑦ \end{array}\right.$

Then by $(-3) \times$ ⑤ $+$ ⑦ we obtain : $(S_5) \quad \left\{\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & - & 3x_3 & = & 4 & — ⑥ \\ & & x_2 & - & 2x_3 & = & 1 & — ⑤ \\ & & & & 0 & = & 0 & — ⑧ \end{array}\right.$

Then by $(-2) \times$ ⑤ $+$ ⑥ we obtain : $(S_6) \quad \left\{\begin{array}{rcrcrcrl} x_1 & & & + & x_3 & = & 2 & — ⑨ \\ & & x_2 & - & 2x_3 & = & 1 & — ⑤ \\ & & & & 0 & = & 0 & — ⑧ \end{array}\right.$

This tells us that the respective values of x_1, x_2 depend on that of x_3 according to the relations $x_1 = 2 - x_3$ and $x_2 = 1 + 2x_3$.

Hence we conclude: *it is possible for* (x_1, x_2, x_3) *to be* $(2-t, 1+2t, t)$ *for some real number* t*, and there is no other possibility.*

Step 2. Checking 'candidate solution'.

[Ask. Suppose $x_1 = 2 - t$ and $x_2 = 1 + 2t$ and $x_3 = t$ for some real number t. Is it true that $x_2 - 2x_3 = 1$ and $-x_1 - 2x_2 + 3x_3 = -4$ and $2x_1 + 7x_2 - 12x_3 = 11$ simultaneously?]

When $x_1 = 2 - t$ and $x_2 = 1 + 2t$ and $x_3 = t$ for some real number t, we have:

$x_2 - 2x_3 = (1 + 2t) - 2t = 1$, and

$-x_1 - 2x_2 + 3x_3 = -(2 - t) - 2(1 + 2t) + 3t = -4$, and

$2x_1 + 7x_2 - 12x_3 = 2(2 - t) + 7(1 + 2t) - 12t = 11$.

Step 3. Drawing conclusion.

The solutions of the system of equations $\left\{\begin{array}{rcrcrcl} & & x_2 & - & 2x_3 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 \end{array}\right.$ is given by $(x_1, x_2, x_3) = (2 - t, 1 + 2t, t)$ where t is an arbitrary real number.

(b) For this problem, it will transpire (after we learn a bit more theory) that when we are only required to determine the solutions of the system, then (with the theoretical results already covering the reasoning in this three-step process,) it will suffice to present as below:

Label the system to be solved by (S_1).

$$(S_1)\quad \left\{\begin{array}{rcrcrcrcc} & & x_2 & - & 2x_3 & = & 1 & — & ① \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 & — & ② \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 & — & ③ \end{array}\right.$$

$$① \leftrightarrow ② : \quad (S_2)\quad \left\{\begin{array}{rcrcrcrcc} -x_1 & - & 2x_2 & + & 3x_3 & = & -4 & — & ④ \\ & & x_2 & - & 2x_3 & = & 1 & — & ⑤ \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 & — & ③ \end{array}\right.$$

$$\vdots \quad \vdots$$

$$(-2) \times ⑤ + ⑥ : \quad (S_6)\quad \left\{\begin{array}{rcrcrcrcc} x_1 & & & + & x_3 & = & 2 & — & ⑨ \\ & & x_2 & - & 2x_3 & = & 1 & — & ⑤ \\ & & & & 0 & = & 0 & — & ⑧ \end{array}\right.$$

The solutions of (S_1) *are given by* $(x_1, x_2, x_3) = (2 - t, 1 + 2t, t)$ *where* t *is an arbitrary real number.*

5. **Example (5).**

What are the solutions of the system of equations $\left\{\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & x_4 & = & 7 \\ x_1 & + & x_2 & + & x_3 & - & x_4 & = & 3 \\ 3x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 1 \end{array}\right.$ *with unknowns* x_1, x_2, x_3, x_4 *in the reals, if there is any at all?*

(a) We may proceed as in the other examples:

- *Step 1. Searching for 'candidate solution' for the system of equations.*
  Suppose (x_1, x_2, x_3, x_4) is a solution of the system concerned.
  After some work, we deduce that $x_1 + 2x_3 - 3x_4 = -1$ and $x_2 - x_3 + 2x_4 = 4$.
  Then it is possible for $(x_1, x_2, x_3, x_4) = (-1 - 2s + 3t, 4 + s - 2t, s, t)$ for some real numbers s, t, and there is no other possibility.
- *Step 2. Checking 'candidate solution'.*
  Suppose $(x_1, x_2, x_3, x_4) = (-1 - 2s + 3t, 4 + s - 2t, s, t)$ for some real numbers s, t.
  Then (x_1, x_2, x_3, x_4) is indeed a solution of the system concerned.
- *Step 3. Drawing conclusion.*
  The solutions of the system concerned is given by $(x_1, x_2, x_3, x_4) = (-1 - 2s + 3t, 4 + s - 2t, s, t)$ where s, t are arbitrary numbers.

(b) It suffices for us to present the manipulations for *Step 1.* Then we may present the description of the solutions as stated in *Step 3.*
Label the system concerned by (S_1).

$$(S_1) \quad \left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & & & + & x_4 & = & 7 & — & ① \\ x_1 & + & x_2 & + & x_3 & - & x_4 & = & 3 & — & ② \\ 3x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 1 & — & ③ \end{array}\right.$$

$$(-1)\times ① + ② : \quad (S_2) \quad \left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & & & + & x_4 & = & 7 & — & ① \\ & - & x_2 & + & x_3 & - & 2x_4 & = & -4 & — & ④ \\ 3x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 1 & — & ③ \end{array}\right.$$

$$(-3)\times ① + ③ : \quad (S_3) \quad \left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & & & + & x_4 & = & 7 & — & ① \\ & - & x_2 & + & x_3 & - & 2x_4 & = & -4 & — & ④ \\ & - & 5x_2 & + & 5x_3 & - & 10x_4 & = & -20 & — & ⑤ \end{array}\right.$$

$$(-1)\times ④ : \quad (S_4) \quad \left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & & & + & x_4 & = & 7 & — & ① \\ & & x_2 & - & x_3 & + & 2x_4 & = & 4 & — & ⑥ \\ & - & 5x_2 & + & 5x_3 & - & 10x_4 & = & -20 & — & ⑤ \end{array}\right.$$

$$5\times ⑥ + ⑤ : \quad (S_5) \quad \left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & & & + & x_4 & = & 7 & — & ① \\ & & x_2 & - & x_3 & + & 2x_4 & = & 4 & — & ⑥ \\ & & & & & & 0 & = & 0 & — & ⑦ \end{array}\right.$$

$$(-2)\times ⑥ + ① : \quad (S_6) \quad \left\{\begin{array}{rcrcrcrcl} x_1 & & & + & 2x_3 & - & 3x_4 & = & -1 & — & ⑧ \\ & & x_2 & - & x_3 & + & 2x_4 & = & 4 & — & ⑥ \\ & & & & & & 0 & = & 0 & — & ⑦ \end{array}\right.$$

The solutions of (S_1) is given by $(x_1, x_2, x_3, x_4) = (-1 - 2s + 3t, 4 + s - 2t, s, t)$ where s, t are arbitrary numbers.

6. **Example (6).**

What are the solutions of the system of equations $\left\{\begin{array}{rcrcrcrcrcl} & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \\ -2x_1 & - & x_2 & - & 3x_3 & + & 3x_4 & + & x_5 & = & 3 \end{array}\right.$ *with unknowns* x_1, x_2, x_3, x_4, x_5 *in the reals, if there is any at all?*

(a) We may proceed as in the other examples:

- *Step 1. Searching for 'candidate solution' for the system of equations.*
  Suppose $(x_1, x_2, x_3, x_4, x_5)$ is a solution of the system concerned.
  After some work, we deduce that $x_1 + x_3 + x_5 = 10$ and $x_2 + x_3 = -8$ and $x_4 + x_5 = 5$.
  Then it is possible for $(x_1, x_2, x_3, x_4, x_5) = (10 - s - t, -8 - s, s, 5 - t, t)$ for some real numbers s, t, and there is no other possibility.
- *Step 2. Checking 'candidate solution'.*
  Suppose $(x_1, x_2, x_3, x_4, x_5) = (10 - s - t, -8 - s, s, 5 - t, t)$ for some real numbers s, t.
  Then $(x_1, x_2, x_3, x_4, x_5)$ is indeed a solution of the system concerned.
- *Step 3. Drawing conclusion.*
  The solutions of the system concerned is given by $(x_1, x_2, x_3, x_4, x_5) = (10 - s - t, -8 - s, s, 5 - t, t)$ where s, t are arbitrary numbers.

(b) It suffices for us to present the manipulations for *Step 1.* Then we may present the description of the solutions as stated in *Step 3.*

Label the system concerned by (S_1).

$$(S_1)\quad \left\{\begin{array}{rcrcrcrcrcl} & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \quad \text{---} \quad \textcircled{1} \\ x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \quad \text{---} \quad \textcircled{2} \\ -2x_1 & - & x_2 & - & 3x_3 & + & 3x_4 & + & x_5 & = & 3 \quad \text{---} \quad \textcircled{3} \end{array}\right.$$

$$\textcircled{1} \leftrightarrow \textcircled{2}: \quad (S_2)\quad \left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \quad \text{---} \quad \textcircled{4} \\ & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \quad \text{---} \quad \textcircled{5} \\ -2x_1 & - & x_2 & - & 3x_3 & + & 3x_4 & + & x_5 & = & 3 \quad \text{---} \quad \textcircled{3} \end{array}\right.$$

$$2 \times \textcircled{4} + \textcircled{3}: \quad (S_3)\quad \left\{\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \quad \text{---} \quad \textcircled{4} \\ & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \quad \text{---} \quad \textcircled{5} \\ & & 3x_2 & + & 3x_3 & + & 7x_4 & + & 7x_5 & = & 11 \quad \text{---} \quad \textcircled{6} \end{array}\right.$$

$$(-3) \times \textcircled{5} + \textcircled{6}: \quad (S_4)\quad \left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \quad \text{---} \quad \textcircled{4} \\ & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \quad \text{---} \quad \textcircled{5} \\ & & & & & & x_4 & + & x_5 & = & 5 \quad \text{---} \quad \textcircled{7} \end{array}\right.$$

$$(-2) \times \textcircled{5} + \textcircled{4}: \quad (S_5)\quad \left\{\begin{array}{rrcrcrcrcl} x_1 & & + & x_3 & - & 2x_4 & - & x_5 & = & 0 \quad \text{---} \quad \textcircled{8} \\ & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \quad \text{---} \quad \textcircled{5} \\ & & & & & x_4 & + & x_5 & = & 5 \quad \text{---} \quad \textcircled{7} \end{array}\right.$$

$$2 \times \textcircled{7} + \textcircled{8}: \quad (S_6)\quad \left\{\begin{array}{rrcrcrcrcr} x_1 & & + & x_3 & & & + & x_5 & = & 10 \quad \text{---} \quad \textcircled{9} \\ & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \quad \text{---} \quad \textcircled{5} \\ & & & & & x_4 & + & x_5 & = & 5 \quad \text{---} \quad \textcircled{7} \end{array}\right.$$

$$(-2) \times \textcircled{7} + \textcircled{5}: \quad (S_7)\quad \left\{\begin{array}{rrcrrcrcr} x_1 & & + & x_3 & & + & x_5 & = & 10 \quad \text{---} \quad \textcircled{9} \\ & x_2 & + & x_3 & & & & = & -8 \quad \text{---} \quad \textcircled{10} \\ & & & & x_4 & + & x_5 & = & 5 \quad \text{---} \quad \textcircled{7} \end{array}\right.$$

The solutions of (S) is given by $(x_1, x_2, x_3, x_4, x_5) = (10 - s - t, -8 - s, s, 5 - t, t)$ where s, t are arbitrary numbers.

7. **Gaussian elimination for systems of linear equations, introduced through an example.**

Consider the system of linear equations

$$(S)\quad \left\{\begin{array}{rcrcrcrcrcrcr} & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 \\ -x_1 & + & 2x_2 & + & x_3 & - & x_4 & & & - & 2x_6 & = & 0 \\ 2x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & + & 2x_5 & + & x_6 & = & 5 \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 \end{array}\right.$$

We are going to determine all the solutions of (S) through a methodical application of these 'equation operations':—

- 'adding a scalar multiple of one equation to another',
- 'multiplying a non-zero scalar to one equation',
- 'interchanging two equations',

to obtain successive systems of linear equations, each having the same solutions as the others, until we reach some system (known as a 'reduced row echelon form' for (S)) from which we can read off all its solutions, and hence that of (S) also.

Such a methodical application of 'equation operations' is referred to as the method of Gaussian elimination.

(a) We start by labelling (S) as (S_1), and labelling all the equations in (S).

We apply appropriate equation operations to obtain a system in which the 'top equation' reads as '$1x_1 + \cdots = \cdots$':

$$(S_1)\quad \left\{\begin{array}{rcrcrcrcrcrcrcc} & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 & \text{---} & \textcircled{1} \\ -x_1 & + & 2x_2 & + & x_3 & - & x_4 & & & - & 2x_6 & = & 0 & \text{---} & \textcircled{2} \\ 2x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & + & 2x_5 & + & x_6 & = & 5 & \text{---} & \textcircled{3} \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 & \text{---} & \textcircled{4} \end{array}\right.$$

$$\textcircled{1} \leftrightarrow \textcircled{2}: \quad (S_2)\quad \left\{\begin{array}{rcrcrcrcrcrcrcc} -x_1 & + & 2x_2 & + & x_3 & - & x_4 & & & - & 2x_6 & = & 0 & \text{---} & \textcircled{5} \\ & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 & \text{---} & \textcircled{6} \\ 2x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & + & 2x_5 & + & x_6 & = & 5 & \text{---} & \textcircled{3} \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 & \text{---} & \textcircled{4} \end{array}\right.$$

$$(-1) \times \textcircled{5}: \quad (S_3)\quad \left\{\begin{array}{rcrcrcrcrcrcrcc} x_1 & - & 2x_2 & - & x_3 & + & x_4 & & & + & 2x_6 & = & 0 & \text{---} & \textcircled{7} \\ & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 & \text{---} & \textcircled{6} \\ 2x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & + & 2x_5 & + & x_6 & = & 5 & \text{---} & \textcircled{3} \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 & \text{---} & \textcircled{4} \end{array}\right.$$

(b) With the 'top equation' in (S_3), we get rid of the x_1's in all other equations by applying the appropriate 'equation operations':

$$(S_3)\quad \left\{\begin{array}{rcrcrcrcrcrcrcc} x_1 & - & 2x_2 & - & x_3 & + & x_4 & & & + & 2x_6 & = & 0 & \text{---} & \textcircled{7} \\ & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 & \text{---} & \textcircled{6} \\ 2x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & + & 2x_5 & + & x_6 & = & 5 & \text{---} & \textcircled{3} \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 & \text{---} & \textcircled{4} \end{array}\right.$$

$$(-2) \times \textcircled{7} + \textcircled{3}: \quad (S_4)\quad \left\{\begin{array}{rcrcrcrcrcrcrcc} x_1 & - & 2x_2 & - & x_3 & + & x_4 & & & + & 2x_6 & = & 0 & \text{---} & \textcircled{7} \\ & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 & \text{---} & \textcircled{6} \\ & & & & x_3 & + & x_4 & + & 2x_5 & - & 3x_6 & = & 5 & \text{---} & \textcircled{8} \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 & \text{---} & \textcircled{4} \end{array}\right.$$

$$(-3) \times \textcircled{7} + \textcircled{4}: \quad (S_5)\quad \left\{\begin{array}{rcrcrcrcrcrcrcc} x_1 & - & 2x_2 & - & x_3 & + & x_4 & & & + & 2x_6 & = & 0 & \text{---} & \textcircled{7} \\ & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 & \text{---} & \textcircled{6} \\ & & & & x_3 & + & x_4 & + & 2x_5 & - & 3x_6 & = & 5 & \text{---} & \textcircled{8} \\ & & & & 2x_3 & + & 2x_4 & + & 4x_5 & - & 6x_6 & = & 10 & \text{---} & \textcircled{9} \end{array}\right.$$

(c) Consider the system (S_5). Leave the 'top equation' in alone for now.

We apply appropriate equation operations to obtain a system in which the 'second top equation' reads as '$1x_j + \cdots = \cdots$', in which j is the smallest index for the unknowns explicitly involved in the equations below the top equation in (S_5).

Then, with the 'second top equation' in the resultant system, we get rid of the x_j's in all other equations.

Here $j = 3$:

$$(S_5)\quad \begin{cases} x_1 - 2x_2 - x_3 + x_4 + 2x_6 = 0 & \text{——}\ \textcircled{7} \\ 2x_3 + 3x_4 + 5x_5 - 7x_6 = 12 & \text{——}\ \textcircled{6} \\ x_3 + x_4 + 2x_5 - 3x_6 = 5 & \text{——}\ \textcircled{8} \\ 2x_3 + 2x_4 + 4x_5 - 6x_6 = 10 & \text{——}\ \textcircled{9} \end{cases}$$

$$\textcircled{6} \leftrightarrow \textcircled{8}: \quad (S_6)\quad \begin{cases} x_1 - 2x_2 - x_3 + x_4 + 2x_6 = 0 & \text{——}\ \textcircled{7} \\ x_3 + x_4 + 2x_5 - 3x_6 = 5 & \text{——}\ \textcircled{10} \\ 2x_3 + 3x_4 + 5x_5 - 7x_6 = 12 & \text{——}\ \textcircled{11} \\ 2x_3 + 2x_4 + 4x_5 - 6x_6 = 10 & \text{——}\ \textcircled{9} \end{cases}$$

$$(-2) \times \textcircled{10} + \textcircled{11}: \quad (S_7)\quad \begin{cases} x_1 - 2x_2 - x_3 + x_4 + 2x_6 = 0 & \text{——}\ \textcircled{7} \\ x_3 + x_4 + 2x_5 - 3x_6 = 5 & \text{——}\ \textcircled{10} \\ x_4 + x_5 - x_6 = 2 & \text{——}\ \textcircled{12} \\ 2x_3 + 2x_4 + 4x_5 - 6x_6 = 10 & \text{——}\ \textcircled{9} \end{cases}$$

$$(-2) \times \textcircled{10} + \textcircled{9}: \quad (S_8)\quad \begin{cases} x_1 - 2x_2 - x_3 + x_4 + 2x_6 = 0 & \text{——}\ \textcircled{7} \\ x_3 + x_4 + 2x_5 - 3x_6 = 5 & \text{——}\ \textcircled{10} \\ x_4 + x_5 - x_6 = 2 & \text{——}\ \textcircled{12} \\ 0 = 0 & \text{——}\ \textcircled{13} \end{cases}$$

(d) Consider the system (S_8). Leave the 'top two equations' alone for now.
We apply appropriate equation operations to obtain a system in which the 'third top equation' reads as '$1x_j + \cdots = \cdots$', in which j is the smallest index for the unknowns explicitly involved in the equations below the 'top two equations' in (S_8).
Then, with the 'third top equation' in the resultant system, we get rid of the x_j's in all other equations. And so forth and so on.
For the equations below the 'top two' in (S_8), $j = 4$:

$$(S_8)\quad \begin{cases} x_1 - 2x_2 - x_3 + x_4 + 2x_6 = 0 & \text{——}\ \textcircled{7} \\ x_3 + x_4 + 2x_5 - 3x_6 = 5 & \text{——}\ \textcircled{10} \\ x_4 + x_5 - x_6 = 2 & \text{——}\ \textcircled{12} \\ 0 = 0 & \text{——}\ \textcircled{13} \end{cases}$$

We observe we need to do nothing with the equations below the 'top two', because the system (S_8) is already in the form we hope to obtain with the 'equation operations' described.

(e) Consider the system (S_8).
We apply appropriate equation operations to obtain a system in which the 'leading unknown' (the unknown with 'smallest index') in each equation will not be involved explicitly in the equations above it:

$$(S_8)\quad \begin{cases} x_1 - 2x_2 - x_3 + x_4 + 2x_6 = 0 & \text{——}\ \textcircled{7} \\ x_3 + x_4 + 2x_5 - 3x_6 = 5 & \text{——}\ \textcircled{10} \\ x_4 + x_5 - x_6 = 2 & \text{——}\ \textcircled{12} \\ 0 = 0 & \text{——}\ \textcircled{13} \end{cases}$$

$$1 \times \textcircled{10} + \textcircled{7}: \quad (S_9)\quad \begin{cases} x_1 - 2x_2 + 2x_4 + 2x_5 - x_6 = 5 & \text{——}\ \textcircled{14} \\ x_3 + x_4 + 2x_5 - 3x_6 = 5 & \text{——}\ \textcircled{10} \\ x_4 + x_5 - x_6 = 2 & \text{——}\ \textcircled{12} \\ 0 = 0 & \text{——}\ \textcircled{13} \end{cases}$$

$$(-2) \times \textcircled{12} + \textcircled{14}: \quad (S_{10})\quad \begin{cases} x_1 - 2x_2 + x_6 = 1 & \text{——}\ \textcircled{15} \\ x_3 + x_4 + 2x_5 - 3x_6 = 5 & \text{——}\ \textcircled{10} \\ x_4 + x_5 - x_6 = 2 & \text{——}\ \textcircled{12} \\ 0 = 0 & \text{——}\ \textcircled{13} \end{cases}$$

$$(-1) \times \textcircled{12} + \textcircled{10}: \quad (S_{11})\quad \begin{cases} x_1 - 2x_2 + x_6 = 1 & \text{——}\ \textcircled{15} \\ x_3 + x_5 - 2x_6 = 3 & \text{——}\ \textcircled{16} \\ x_4 + x_5 - x_6 = 2 & \text{——}\ \textcircled{12} \\ 0 = 0 & \text{——}\ \textcircled{13} \end{cases}$$

(f) From the system (S_{11}), we obtain the relations

$$\begin{cases} x_1 = 1 + 2x_2 - x_6 \\ x_3 = 3 - x_5 + 2x_6 \\ x_4 = 2 - x_5 + x_6 \end{cases}$$

The solutions of the system (S) is described by $(x_1, x_2, x_3, x_4, x_5, x_6) = (1 + 2r - t, r, 3 - s + 2t, 2 - s + t, s, t)$, where r, s, t are arbitrary numbers.

Remark. Throughout Examples (1), (2), (3), (4), (5), (6), we have been using Gaussian elimination to determine all solutions for the respective systems; the actual manipulations are done when we search for 'candidate solutions', which turn out to be the solutions.