THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2022-2023 Term 1
Homework Assignment 3
Suggested Solutions for HW3

1. By using Lagrange's mean value theorem, or otherwise, show that

   (a) $\sin x \le x$ for all $x \in [0, +\infty)$.

   (b) $(1+x)^p \ge 1 + px$ for any $p \ge 1$ and $x \ge 0$.

   Solution:

   (a) Suppose $f(x) = \sin x, f'(x) = \cos x$, then $f(x)$ is continuous on $[0, +\infty)$ and differentiable on $(0, +\infty)$.
   By Lagrange's Mean Value Theorem, $\forall x_0 \in (0, +\infty), \exists \xi \in (0, x_0), \text{s.t.} f'(\xi) = \dfrac{f(x_0) - f(0)}{x_0 - 0}$.Thus $\dfrac{\sin x_0}{x_0} = \cos\xi \in [-1, 1]$ which means $\sin x_0 \le x_0$. More, if $x_0 = 0, \sin x_0 = 0 = x_0$. So $\sin x \le x,\ \forall x \in [0, +\infty)$.

   (b) Suppose $f(x) = (1+x)^p, f'(x) = p(1+x)^{(p-1)}$, then $f(x)$ is continuous on $[0, +\infty)$ and differentiable on $(0, +\infty)$ since $p \ge 1$.
   By Lagrange's Mean Value Theorem, $\forall x_0 \in (0, +\infty), \exists \xi \in (0, x_0), \text{s.t.} f'(\xi) = \dfrac{f(x_0) - f(0)}{x_0 - 0}$.Thus $\dfrac{(1+x_0)^p - 1}{x_0} = p(1+\xi)^{(p-1)} \ge p$ which means $(1+x_0)^p \ge 1 + px_0$. More, if $x_0 = 0, (1+x_0)^p = 1 = 1 + px_0$. So $(1+x)^p \ge 1 + px,\ \forall x \in [0, +\infty)$.

2. Let $0 < a < b < \dfrac{\pi}{2}$. Prove that there exists $a < \xi < b$ such that

$$\ln\left(\frac{\cos a}{\cos b}\right) = (b-a)\tan\xi.$$

   Solution: Suppose $f(x) = \ln\cos x, f'(x) = -\dfrac{\sin x}{\cos x} = -\tan x$, then $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) since $0 < a < b < \dfrac{\pi}{2}$.
   By Lagrange's Mean Value Theorem, $\exists \xi \in (a, b), \text{s.t.} f'(\xi) = \dfrac{f(a) - f(b)}{a - b}$.Thus $\dfrac{\ln\cos a - \ln\cos b}{a - b} = -\tan\xi$ which means $\ln\dfrac{\cos a}{\cos b} = (b-a)\tan\xi$.

3. Show that for all $0 < a < b \le 1$,

$$(b-a)(1+\ln a) < \ln\left(\frac{b^b}{a^a}\right) < (b-a)(1+\ln b).$$

   Solution: Let $f(x) = x\ln x$ for $x > 0$. Consider $0 < a < b$, we know that f is continuous on $[a, b]$ and differentiable on (a, b). By the Lagrange's mean value theorem, there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = \frac{b\ln b - a\ln a}{b - a} = f^{'}(c) = 1 + \ln c.$$

Since $a < c < b$, $\ln a < \ln c < \ln b$. Thus

$$1 + \ln a < \frac{b \ln b - a \ln a}{b - a} < 1 + \ln b.$$

That is,

$$(b-a)(1 + \ln a) < \ln(\frac{b^b}{a^a}) < (b-a)(1 + \ln b).$$

4. Evaluate the following limits.

(a) $\lim_{x \to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$

(b) $\lim_{x \to 0} \log_{\tan x}(\tan 2x)$

(c) $\lim_{x \to 0^+} \tan x \ln \sin x$

(d) $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{x}{x-1} \right)$

(e) $\lim_{x \to +\infty} \frac{e^{1 + \ln x}}{\ln(1 + e^x)}$

Solution:

(a) We compute the Taylor series of $\sin^{-1} x$ and $\tan^{-1} x$ at $x = 0$ to the third order:

$$(\sin^{-1})'(x) = (1 - x^2)^{\frac{-1}{2}}$$
$$(\sin^{-1})''(x) = x(1 - x^2)^{\frac{-3}{2}}$$
$$(\sin^{-1})'''(x) = (1 + 2x^2)(1 - x^2)^{\frac{-5}{2}}$$

$$(\tan^{-1})'(x) = (1 + x^2)^{-1}$$
$$(\tan^{-1})''(x) = -2x(1 + x^2)^{-2}$$
$$(\tan^{-1})'''(x) = -2(1 - x^2)(1 + x^2)^{-3}$$

So the Taylor series are

$$\sin^{-1}(x) = x + \frac{x^3}{6} + O(x^4)$$

and

$$\tan^{-1}(x) = x - \frac{x^3}{3} + O(x^4)$$

Hence the limit is

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3} &= \lim_{x \to 0} \frac{(x + \frac{1}{6}x^3 + O(x^4)) - (x - \frac{1}{3}x^3 + O(x^4))}{x^3} \\
&= \lim_{x \to 0} \frac{\frac{1}{2}x^3 + O(x^4)}{x^3} \\
&= \frac{1}{2}
\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0}\log_{\tan x}(\tan 2x) &= \lim_{x\to 0}\frac{\ln\tan 2x}{\ln\tan x}\\ &= \lim_{x\to 0}\frac{(\ln\tan 2x)'}{(\ln\tan x)'}\\ &= \lim_{x\to 0} 2\frac{\tan x\cos^2 x}{\tan 2x\cos^2 2x}\\ &= \lim_{x\to 0} 2\frac{\sin 2x}{\sin 4x}\\ &= \lim_{x\to 0}\frac{\sin 2x}{2x}\lim_{x\to 0}\frac{4x}{\sin 4x}\\ &= 1\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0^+}\tan x\ln\sin x &= \lim_{x\to 0^+}\frac{\ln\sin x}{\tan x}\\ &= \lim_{x\to 0^+}\frac{(\ln\sin x)'}{(\tan x)'}\\ &= \lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cos x}{\frac{-1}{\tan^2 x}\sec^2 x}\\ &= \lim_{x\to 0^+} -\sin x\cos x = 0\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right) &= \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}-1\\ &= \lim_{x\to 1}\frac{(x-1-\ln x)'}{((x-1)\ln x)'}-1\\ &= \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}-1\\ &= \lim_{x\to 1}\frac{x-1}{x\ln x+x-1}-1\\ &= \lim_{x\to 1}\frac{(x-1)'}{(x\ln x+x-1)'}-1\\ &= \lim_{x\to 1}\frac{1}{\ln x+2}-1=-\frac{1}{2}\end{aligned}$$

(e)

$$\begin{aligned}\lim_{x\to+\infty}\frac{e^{1+\ln x}}{\ln(1+e^x)} &= \lim_{x\to+\infty}\frac{xe}{\ln(1+e^x)}\\ &= \lim_{x\to+\infty}\frac{(xe)'}{(\ln(1+e^x))'}\\ &= \lim_{x\to+\infty}\frac{e}{\frac{1}{1+e^x}e^x}\\ &= \lim_{x\to+\infty} e(1+e^{-x}) = e\end{aligned}$$

5. Evaluate the following limits.

(a) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(b) $\lim_{x\to 1} x^{\frac{2x}{x-1}}$

(c) $\lim_{x\to 0}\frac{(1+x)^x-1}{x^2}$

(d) $\lim_{x\to+\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$

Solution:

(a)

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x^2}\ln\frac{\sin x}{x} &= \lim_{x\to 0}\frac{(\ln\frac{\sin x}{x})'}{(x^2)'}\\
&= \lim_{x\to 0}\frac{\frac{x}{\sin x}(\frac{\cos x}{x}-\frac{\sin x}{x^2})}{2x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{x\cos x-\sin x}{x^2\sin x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{(x\cos x-\sin x)'}{(x^2\sin x)'}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{-x\sin x}{2x\sin x+x^2\cos x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{-1}{2+\frac{x}{\tan x}}\\
&= \frac{1}{2}\frac{-1}{2+1}=-\frac{1}{6}
\end{aligned}$$

So

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} &= e^{\lim_{x\to 0}\frac{1}{x^2}\ln\frac{\sin x}{x}}\\
&= e^{\frac{-1}{6}}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 1}\frac{2x}{x-1}\ln x &= 2\lim_{x\to 1}\frac{\ln x}{1-\frac{1}{x}}\\
&= 2\lim_{x\to 1}\frac{(\ln x)'}{(1-\frac{1}{x})'}\\
&= 2\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x^2}}=2
\end{aligned}$$

So

$$\begin{aligned}
\lim_{x\to 1} x^{\frac{2x}{x-1}} &= e^{\lim_{x\to 1}\frac{2x}{x-1}\ln x}\\
&= e^2
\end{aligned}$$

(c) We compute the Taylor series of $f(x) = (1+x)^x = e^{x\ln(1+x)}$ at $x = 0$ up to x^2:

$$f'(x) = e^{x\ln(1+x)}(\ln(1+x) + 1 - \frac{1}{1+x})$$

$$f''(x) = e^{x\ln(1+x)}(\ln(1+x) + 1 - \frac{1}{1+x})^2 + e^{x\ln(1+x)}\frac{x+2}{(1+x)^2}$$

As $f(0) = e^{0\ln 1} = 1$, $f'(0) = e^{0\ln 1}(\ln 1 + 1 - \frac{0}{1+0}) = 0$, $f''(0) = e^{0\ln 1}(\ln 1 + 1 - \frac{0}{1+0})^2 + e^{0\ln 1}\frac{0+2}{(1+0)^2} = 2$, we have $(1+x)^x = 1 + x^2 + O(x^3)$, so

$$\lim_{x\to 0}\frac{(1+x)^x - 1}{x^2} = \lim_{x\to 0}\frac{x^2 + O(x^3)}{x^2} = 1$$

(d)

$$\begin{aligned}\lim_{x\to+\infty} x\ln\frac{(x-1)^2}{x^2-4x+2} &= \lim_{x\to+\infty}\frac{\ln\frac{(x-1)^2}{x^2-4x+2}}{x^{-1}} \\ &= \lim_{x\to+\infty}\frac{(\ln\frac{(x-1)^2}{x^2-4x+2})'}{(x^{-1})'} \\ &= \lim_{x\to+\infty}\frac{2}{-x^{-2}}\frac{-x}{(x-1)(x^2-4x+2)} = 2\end{aligned}$$

So

$$\lim_{x\to+\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x = e^{\lim\limits_{x\to+\infty} x\ln\frac{(x-1)^2}{x^2-4x+2}} = e^2$$

6. For each of the following functions $f(x)$, find

- domain of f and x, y-intercepts
- asymptotes of $y = f(x)$
- $f'(x)$, local maximum, local minimum, intervals on which f is increasing, decreasing
- $f'(x)$, points of inflection, intervals on which f is concave up, down

Then sketch the graph of $y = f(x)$.

(a) $f(x) = \dfrac{x}{(x-2)^2}$

(b) $f(x) = \dfrac{x^2+5x+7}{x+2}$

(c) $f(x) = \dfrac{x^2}{x^2-2x+2}$

(d) $f(x) = x^{\frac{2}{3}} - 1$

Solution:

(a)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{(x-2)^2} = \frac{1}{(x-2)^2} - \frac{2x}{(x-2)^3} = -\frac{x+2}{(x-2)^3}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} - \frac{x+2}{(x-2)^3} = -\left(\frac{1}{(x-2)^3} - \frac{3(x+2)}{(x-2)^4}\right) = \frac{2x+8}{(x-2)^4}$$

f is differentiable on the domain $(-\infty, 2) \cup (2, \infty)$, and $f'(x) > 0$ if and only if $-2 < x < 2$. So f is increasing on $[-2, 2)$.

Since when $x = 2$, the denominator becomes 0, so $x = 2$ is an vertical asymptote.

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ and $\lim_{x\to\pm\infty} f(x) = 0$, $y = 0$ is an asymptote.

The only critical point of $f(x)$ is $x = -2$, at which $f''(-2) = \frac{1}{64} > 0$, so $x = -2$ is the only relative extremum and is a relative minimum.

(b)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+5x+7}{x+2} = \frac{2x+5}{x+2} - \frac{x^2+5x+7}{(x+2)^2} = \frac{x^2+4x+3}{(x+2)^2} = \frac{(x+1)(x+3)}{(x+2)^2}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+4x+3}{(x+2)^2} = \frac{2x+4}{(x+2)^2} - (x^2+4x+3)\frac{-2}{(x+2)^3} = \frac{2}{(x+2)^3}$$

f is differentiable on the domain $(-\infty, -2) \cup (-2, \infty)$, and $f'(x) > 0$ if and only if $x < -3$ or $-1 < x$. Also, $f(-3) = -1 < 3 = f(-1)$. So f is increasing on $(-\infty, -3] \cup [-1, \infty)$

Since when $x = -2$, the denominator becomes 0, so $x = -2$ is an vertical asymptote.

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 1$ and $\lim_{x\to\pm\infty} f(x) - x = 3$, so $y = x + 3$ is an asymptote.

The only critical points are $x = -1$ and $x = -3$. Since $f''(-1) = 2 > 0$ and $f''(-3) = -2 < 0$, so the only relative extrema are at $x = -1$ and $x = -3$, where $x = -1$ is a relative minimum and $x = -3$ is a relative maximum.

(c)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{x^2-2x+2} = \frac{2x}{x^2-2x+2} - \frac{x^2(2x-2)}{(x^2-2x+2)^2} = -\frac{2x(x-2)}{(x^2-2x+2)^2}$$

$$f''(x) = \frac{-4x+4}{(x^2-2x+2)^2} - \frac{2(-2x^2+4x)(2x-2)}{(x^2-2x+2)^3} = \frac{4(x-1)(x^2-2x-2)}{(x^2-2x+2)^3}$$

f is differential on the domain $(-\infty, \infty)$, and $f'(x) > 0$ if and only if $0 < x < 2$, so f is increasing on $[0, 2]$

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ and $\lim_{x\to\pm\infty} f(x) = 1$, $y = 1$ is an asymptote.

The critical points of f are $x = 0$ and $x = 2$. Since $f''(0) = 1 > 0$ and $f''(2) = -1 < 0$, so the only relative extrema are $x = 0$ and $x = 2$, where $x = 0$ is a relative minimum and $x = 2$ is a relative maximum.

(d)

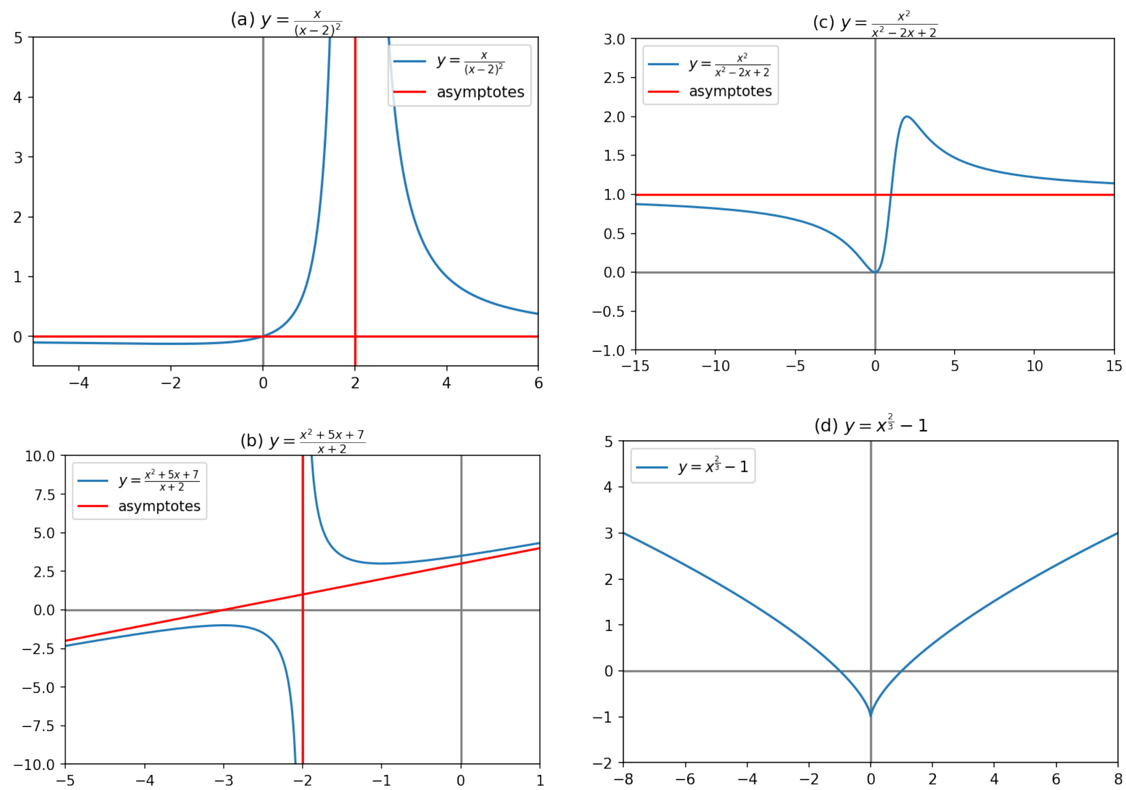
$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{2}{3}} - 1) = \frac{2}{3}x^{\frac{-1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{3}x^{\frac{-1}{3}} = -\frac{2}{9}x^{\frac{-4}{3}} = -\frac{2}{9\sqrt[3]{x^4}}$$

f is differentiable on $(-\infty, 0) \cup (0, \infty)$, and $f'(x) > 0$ if and only if $x > 0$. So f is increasing on $[0, \infty)$

Since $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ but $\lim_{x\to\pm\infty} f(x)$ does not exist. So f has no asymptote.

The only critical points of f are $x = 0$ as f is not differentiable at $x = 0$ and $f'(x) \neq 0$ on $(-\infty, 0) \cup (0, \infty)$. Since for $x \neq 0$, $f(x) = -1 + \sqrt[3]{x^2} \geq -1 = f(0)$, $x = 0$ is the only relative extremum and is a relative minimum.



Remark:
The graphs of the functions for question 6. Asymptotes, if they exist, are also drawn.

7. Find the Taylor series up to the term in $(x - c)^3$ of the functions about $x = c$.

   (a) $\dfrac{1}{1+x}$; $c = 1$.
   (b) $\dfrac{2-x}{3+x}$; $c = 1$.
   (c) $\dfrac{x}{(x-1)(x-2)}$; $c = 0$.
   (d) $\cos x$; $c = \frac{\pi}{4}$.
   (e) $\sin^2 x$; $c = 0$
   (f) $\ln x$; $c = e$.
   (g) 3^x; $c = 0$.
   (h) $\sqrt{2+x}$; $c = 1$.
   (i) $\dfrac{1}{\sqrt{7-3x}}$; $c = 1$.

Solution:

(a) Let $f(x) = \frac{1}{1+x}$. Then $f(c) = \frac{1}{1+c} = \frac{1}{2}$, $f'(c) = \frac{-1}{(1+c)^2} = -\frac{1}{4}$, $f''(c) = \frac{2}{(1+c)^3} = \frac{1}{4}$, $f'''(c) = \frac{-6}{(1+c)^4} = -\frac{3}{8}$.
So $\frac{1}{1+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-c)^3 + O((x-c)^4)$

(b) Let $f(x) = \frac{2-x}{3+x} = -1 + \frac{5}{3+x}$. Then $f(c) = -1 + \frac{5}{3+c} = \frac{1}{4}$, $f'(c) = \frac{-5}{(3+c)^2} = -\frac{5}{16}$, $f''(c) = \frac{10}{(3+c)^3} = \frac{5}{32}$, $f'''(c) = \frac{-30}{(3+c)^4} = -\frac{15}{128}$.

So $\frac{2-x}{3+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= \frac{1}{4} - \frac{5}{16}(x-1) + \frac{5}{64}(x-1)^2 - \frac{5}{256}(x-1)^3 + O((x-1)^4)$

(c) Let $f(x) = \frac{x}{(x-1)(x-2)}$. Then $f(c) = \frac{0}{(0-1)(0-2)} = 0$, $f'(c) = -\frac{c^2-2}{(c-1)^2(c-2)^2} = \frac{1}{2}$, $f''(c) = \frac{2(c^3-6c+6)}{(c-1)^3(c-2)^3} = \frac{3}{2}$, $f'''(c) = -\frac{6(c^4-12c^2+24c-14)}{(c-1)^4(c-2)^4} = \frac{21}{4}$.
So $\frac{x}{(x-1)(x-2)} = f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\frac{f'''(c)}{6}(x-c)^3+O((x-c)^4)$
$= \frac{1}{2}x + \frac{3}{4}x^2 + \frac{7}{8}x^3 + O(x^4)$

(d) Let $f(x) = \cos x$. Then $f(c) = \cos c = \frac{\sqrt{2}}{2}$, $f'(c) = -\sin c = -\frac{\sqrt{2}}{2}$, $f''(c) = -\cos c = -\frac{\sqrt{2}}{2}$, $f'''(c) = \sin c = \frac{\sqrt{2}}{2}$.
So $\cos x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 + \frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3 + O((x-\frac{\pi}{4})^4)$

(e) Let $f(x) = \sin^2 x$. Then $f(c) = \sin^2 c = 0$, $f'(c) = \sin(2c) = 0$, $f''(c) = 2\cos(2c) = 2$, $f'''(c) = -4\sin(2c) = 0$.
So $\sin^2 x = f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\frac{f'''(c)}{6}(x-c)^3+O((x-c)^4)$
$= x^2 + O(x^4)$

(f) Let $f(x) = \ln x$. Then $f(c) = \ln c = 1$, $f'(c) = \frac{1}{c} = \frac{1}{e}$, $f''(c) = -\frac{1}{c^2} = -\frac{1}{e^2}$, $f'''(c) = \frac{2}{c^3} = \frac{2}{e^3}$.
So $\ln x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + O((x-e)^4)$

(g) Let $f(x) = 3^x$. Then $f(c) = 3^c = 1$, $f'(c) = 3^c\ln 3 = \ln 3$, $f''(c) = 3^c(\ln 3)^2 = (\ln 3)^2$, $f'''(c) = 3^c(\ln 3)^3 = (\ln 3)^3$.
So $3^x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= 1 + x\ln 3 + \frac{(\ln 3)^2}{2}x^2 + \frac{(\ln 3)^3}{6}x^3 + O(x^4)$

(h) Let $f(x) = \sqrt{2+x}$. Then $f(c) = \sqrt{2+c} = \sqrt{3}$, $f'(c) = \frac{1}{2}(2+c)^{\frac{-1}{2}} = \frac{\sqrt{3}}{6}$, $f''(c) = -\frac{1}{4}(2+c)^{\frac{-3}{2}} = -\frac{\sqrt{3}}{36}$, $f'''(c) = \frac{3}{8}(2+c)^{\frac{-5}{2}} = \frac{\sqrt{3}}{72}$.
So $\sqrt{2+x} = f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\frac{f'''(c)}{6}(x-c)^3+O((x-c)^4)$
$= \sqrt{3} + \frac{\sqrt{3}}{6}(x-1) - \frac{\sqrt{3}}{72}(x-1)^2 + \frac{\sqrt{3}}{432}(x-1)^3 + O((x-1)^4)$

(i) Let $f(x) = \frac{1}{\sqrt{7-3x}}$. Then $f(c) = \frac{1}{\sqrt{7-3c}} = \frac{1}{2}$, $f'(c) = -\frac{1}{2}(7-3c)^{\frac{-3}{2}} = \frac{3}{16}$, $f''(c) = \frac{27}{3}(7-3x)^{\frac{-5}{2}} = \frac{27}{128}$, $f'''(c) = \frac{405}{8}(7-3x)^{\frac{-7}{2}} = \frac{405}{1024}$.
So $\frac{1}{\sqrt{7-3c}} = f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\frac{f'''(c)}{6}(x-c)^3+O((x-c)^4)$
$= \frac{1}{2} + \frac{3}{16}(x-1) + \frac{27}{256}(x-1)^2 + \frac{135}{2048}(x-1)^3 + O((x-1)^4)$

Alternatively, by using the Taylor series of the elementary functions,

(a) $\frac{1}{x+1} = \frac{1}{2}\frac{1}{1+\frac{x-1}{2}} = \frac{1}{2}\left(1 - \frac{x-1}{2} + (\frac{x-1}{2})^2 - (\frac{x-1}{2})^3 + O((x-1)^4)\right)$
$= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + O((x-1)^4)$

(b) $\frac{2-x}{3+x} = -1 + \frac{5}{4}\frac{1}{1+\frac{x-1}{4}} = -1 + \frac{5}{4}\left(1 - \frac{x-1}{4} + (\frac{x-1}{4})^2 - (\frac{x-1}{4})^3 + O((x-1)^4)\right)$
$= \frac{1}{4} - \frac{5}{16}(x-1) + \frac{5}{64}(x-1)^2 - \frac{5}{256}(x-1)^3 + O((x-4)^2)$

(c) $\frac{x}{(x-1)(x-2)} = -\frac{1}{1-\frac{x}{2}} + \frac{1}{1-x}$
$= -\left(1 + \frac{x}{2} + (\frac{x}{2})^2 + (\frac{x}{2})^3 + O(x^4)\right) + (1 + x + x^2 + x^3 + O(x^4))$
$= \frac{1}{2}x + \frac{3}{4}x^2 + \frac{7}{8}x^3 + O(x^4)$

(d) $\cos x = \cos(x - \frac{\pi}{4} + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}\left(\cos(x - \frac{\pi}{4}) - \sin(x - \frac{\pi}{4})\right)$
$= \frac{\sqrt{2}}{2}\left((1 - \frac{(x-\frac{\pi}{4})^2}{2} + O((x - \frac{\pi}{4})^4)) - ((x - \frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^3}{6} + O((x - \frac{\pi}{4})^4))\right)$
$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x - \frac{\pi}{4})^2 + \frac{\sqrt{2}}{12}(x - \frac{\pi}{4})^3 + O((x - \frac{\pi}{4})^4)$

(e) $\sin^2 x = \frac{1}{2}(1 - \cos(2x)) = \frac{1}{2}(1 - (1 - \frac{(2x)^2}{2} + O(x^4)))$
$= x^2 + O(x^4)$

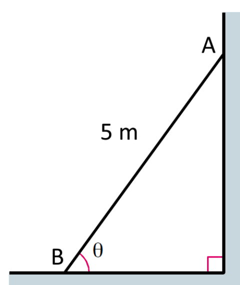
(f) $\ln x = 1 + \ln(1 + \frac{x-e}{e}) = 1 + \left(\frac{x-e}{e} - \frac{1}{2}(\frac{x-e}{e})^2 + \frac{1}{3}(\frac{x-e}{e})^3 + O((x-e)^4)\right)$
$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + O((x-e)^4)$

(g) $3^x = e^{x\ln 3} = 1 + x\ln 3 + \frac{1}{2}(x\ln 3)^2 + \frac{1}{6}(x\ln 3)^3 + O(x^4)$
$= 1 + x\ln 3 + \frac{(\ln)^3}{2}x^2 + \frac{(\ln 3)^3}{6}x^3 + O(x^4)$

(h) $\sqrt{2+x} = \sqrt{3}(1 + \frac{x-1}{3})^{\frac{1}{2}}$
$= \sqrt{3}\left(1 + \frac{1}{2}\frac{x-1}{3} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(\frac{x-1}{3})^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{6}(\frac{x-1}{3})^3 + O((x-1)^4)\right)$
$= \sqrt{3} + \frac{\sqrt{3}}{6}(x-1) - \frac{\sqrt{3}}{72}(x-1)^2 + \frac{\sqrt{3}}{432}(x-1)^3 + O((x-1)^4)$

(i) $\frac{1}{\sqrt{7-3x}} = \frac{1}{2}(1 - \frac{x-1}{4/3})^{\frac{-1}{2}}$
$= \frac{1}{2}(1 - \frac{-1}{2}\frac{x-1}{4/3} + \frac{\frac{-1}{2}(\frac{-1}{2}-1)}{2}(\frac{x-1}{4/3})^2 - \frac{\frac{-1}{2}(\frac{-1}{2}-1)(\frac{-1}{2}-2)}{6})(\frac{x-1}{4/3})^3 + O((x-1)^4)$
$= \frac{1}{2} + \frac{3}{16}(x-1) + \frac{27}{256}(x-1)^2 + \frac{135}{2048}(x-1)^3 + O((x-1)^4)$

8. In the following figure



a ladder with length 5 m leans against a wall. The point of contact A between the ladder and the wall slides down at a constant speed of 0.8 m/s. When A is 4.8 m above the ground,

(a) find the sliding speed of B away from the wall
(b) find the rate of change of θ (in degree/s, correct to 2 decimal places)

Solution:

(a) Let x m be the height of A above the ground, and y m be the distance of B from the wall. Then $y = \sqrt{5^2 - x^2}$ and $\frac{dx}{dt} = -0.8$. Hence,

$$\frac{dy}{dt} = \frac{-2x}{2\sqrt{5^2 - x^2}} \cdot \frac{dx}{dt}.$$

When $x = 4.8$,

$$\frac{dy}{dt} = \frac{-(4.8)}{\sqrt{5^2 - 4.8^2}} \cdot (-0.8) = \frac{96}{35}.$$

Therefore, B is sliding away from the wall at a speed of $\frac{96}{35}$ m/s, when A is 4.8 m above the ground.

(b) Note that

$$x = 5\sin\theta.$$

Differentiate both sides with respect to t, we have

$$\frac{dx}{dt} = 5\cos\theta \cdot \frac{d\theta}{dt} = \sqrt{5^2 - 4.8^2} \cdot \frac{d\theta}{dt}.$$

When $x = 4.8$, we have

$$\frac{d\theta}{dt} = \frac{-0.8}{\sqrt{5^2 - 4.8^2}} = -\frac{4}{7}.$$

Therefore, when A is 4.8 m above the ground, the rate of change of θ is

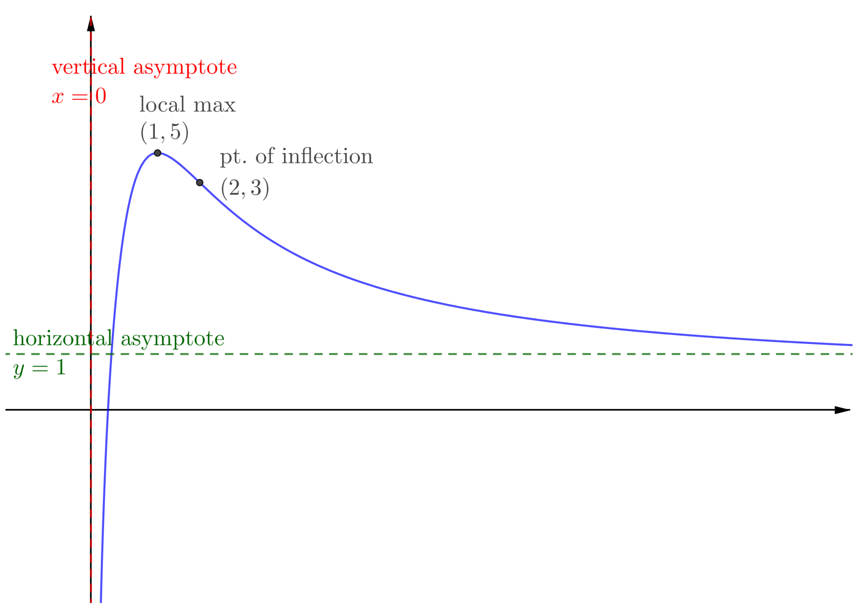
$$-\frac{4}{7} \cdot \frac{180°}{\pi} \approx -32.74°/\text{s}.$$

9. Sketch a graph of a twice-differentiable function $f : (0, \infty) \to \mathbb{R}$ which satisfies the followings:

- $f(1) = 5$ and $f(2) = 3$
- $\lim_{x \to 0^+} f(x) = -\infty$ (DNE) and $\lim_{x \to \infty} f(x) = 1$
- $f'(x) > 0$ over $(0, 1)$ and $f'(x) < 0$ over $(1, \infty)$
- $f''(x) < 0$ over $(0, 2)$ and $f''(x) > 0$ over $(2, \infty)$

On your graph, label any local maximum(s), local maximum(s), point of inflection(s) and asymptote(s) (if any).

Solution:



10. Find the global maximum and minimum (if exist) of

$$f(x) = x^{\frac{4}{5}} e^{-x}$$

with domain $[-1, \infty)$

Solution:

First of all, notice that

$$\begin{aligned} Rf'(0) &= \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0^+} \frac{h^{\frac{4}{5}} e^{-h}}{h} \\ &= \lim_{h \to 0^+} \frac{e^{-h}}{h^{\frac{1}{5}}} = \infty \text{ (DNE)} \end{aligned}$$

So, f is not differentiable at 0. When $x \in [-1, 0) \cup (0, \infty)$,

$$f'(x) = (\frac{4}{5} x^{-\frac{1}{5}} - x^{\frac{4}{5}}) e^{-x}$$

Thus,

$$f'(x) = 0 \iff x = \frac{4}{5}$$

If $x \in [-1, 0) \cup (\frac{4}{5}, +\infty)$, $f'(x) < 0 \implies f$ is strictly decreasing over $[-1, 0]$ and over $[\frac{4}{5}, +\infty)$

Also, if $x \in (0, \frac{4}{5})$, $f'(x) > 0 \implies f$ is strictly increasing over $[0, \frac{4}{5}]$

Since $f(-1) > f(\frac{4}{5})$, $f(x)$ attains its global max value e at $x = -1$. Since $f(0) = 0$ and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^{\frac{4}{5}}}{e^x} = 0,$$

$f(x)$ attains its global min value 0 at $x = 0$.

11. Suppose

$$f(x) = \sqrt{1 + x}$$

(a) Find the Taylor polynomials, $T_n(x)$, of order $n = 0, 1, 2, 3$ of $f(x)$ with center 0.

(b) Use $T_0(x), T_1(x), T_2(x), T_3(x)$ to approximate the value of $\sqrt{1.2}$

Solution:

(a) By definition, the n-th Taylor polynomial $T_n(x)$ of $f(x)$ with center 0 is given by

$$T_n(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n.$$

And we may easily calculate out that

$$
\begin{aligned}
f'(x) &= \frac{1}{2}(1+x)^{-\frac{1}{2}},\\
f''(x) &= -\frac{1}{4}(1+x)^{-\frac{3}{2}},\\
f'''(x) &= \frac{3}{8}(1+x)^{-\frac{5}{2}}.
\end{aligned}
$$

Therefore,

$$f(0) = 1, f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{4}, f'''(0) = \frac{3}{8}$$

So we obtain

$$
\begin{aligned}
T_0(x) &= f(0) = 1,\\
T_1(x) &= f(0) + \frac{f'(0)}{1!}x = 1 + \frac{1}{2}x,\\
T_2(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2,\\
T_3(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.
\end{aligned}
$$

(b) Note that $\sqrt{1.2} = f(0.2)$. So, when using $T_0(x)$,$T_1(x)$,$T_2(x)$,$T_3(x)$ to approximate $f(0.2)$, we have

$$
\begin{aligned}
T_0(0.2) &= 1,\\
T_1(0.2) &= 1 + \frac{1}{2}\cdot 0.2 = 1.1,\\
T_2(0.2) &= 1 + \frac{1}{2}\cdot 0.2 - \frac{1}{8}\cdot (0.2)^2 = 1.095,\\
T_3(0.2) &= 1 + \frac{1}{2}\cdot 0.2 - \frac{1}{8}\cdot (0.2)^2 + \frac{1}{16}\cdot (0.2)^3 = 1.0955.
\end{aligned}
$$

Remark: In fact,

$$\sqrt{1.2} \approx 1.095445115$$

12. Find the exact value of

$$\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

Solution: By Taylor series, since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

with $x = -1$,

$$
\begin{aligned}
e^{-1} &= 1 + \frac{(-1)}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \cdots\\
&= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots\\
&= 1 - \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots\right).
\end{aligned}
$$

Hence,

$$\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots = 1 - \frac{1}{e}.$$