MATH 2550
Notes 1

(Keywords: vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, equations of planes, lines, tangent line to curves, to surfaces, equation of tangent line to a curve, equation(s) of a surface, normal vector, partial derivative, definite integral & area "under $y = f(x)$", "area under $y = f(x)$ as an infinite sum, Riemann Sum, $f(\xi_i)$, Δx_i, line integral of a scalar field, line integral of a vector field, Green's Theorem for a rectangle, Green's Theorem in general, orientation of a curve.)

This set of notes will be very concise. Try to make sure that you understand each and every of the above-mentioned keywords.

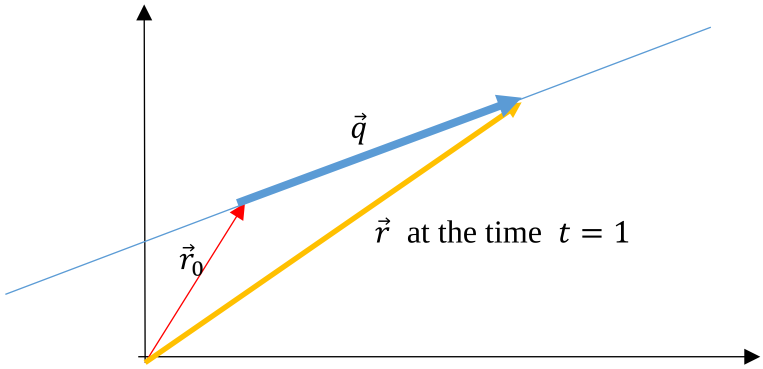
- What is a line in $\mathbb{R}^2$? Ans: Before answering it, we should ask ourselves: "what is the meaning of the question?", or "what the person wants as answer(s)?"
- Depending on how you "read" the question, there are various answers.
- For us, one way to answer it is find way(s) to (specify the (equation) of (each point) on any (given) line in the plane)
- More mathematically, one can say: "a (line) is a (collection) of (points) satisfying (some (special) forms of equations)".
- If you write it down in form of a mathematical (recipe), it is:
  A line in $\mathbb{R}^2$ $= \{\text{point in } \mathbb{R}^2 : \text{point satisfies some equations}\}$
- Note that this is the (grammar) for the (sentence) describing (mathematically) a (line) in $\mathbb{R}^2$**.** (I repeat – it is of the form: (A line in $\mathbb{R}^2$ = Right-hand side, which is enclosed by two curly brackets, i.e. $\{$ and $\}$**.** And inside the curly brackets, the (grammar) is {object **:** properties of the object}, where we (i) first put the (object), followed by (colon), then (ii) we put the (properties of the object).)
- Now comes our example, i.e. a line. Following what we have written down in the previous bullet point, we have

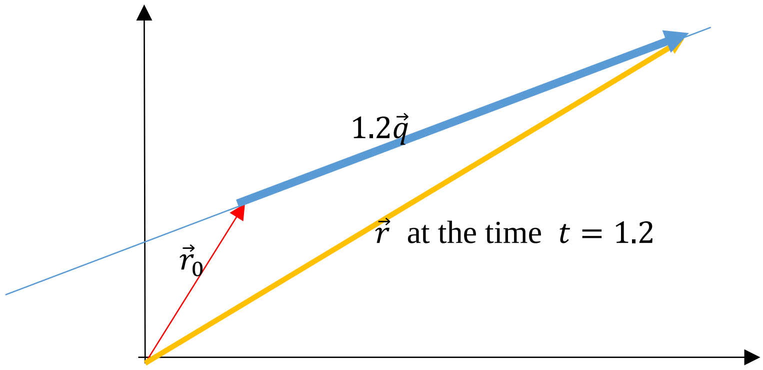
  A line in $\mathbb{R}^2$ = $\{point: \ properties\ satisfied\ by\ this\ point\}$

  Remark: When we use the word (point), we mean (position vector) of a (point) on (the line).

  Question: What are the "properties satisfied by such a point?
  Answer: There are various ways to answer it. One way is to say that such a point (i) starts at some point (again position vector) and runs continuously (along) a certain (direction) [this one isn't (position vector!)

(Picture):





At each (time) t, we get the (position) vector (i.e. the "yellow" vector) of a point, given by the symbol $\vec{r}$ by means of the equation, $\vec{r} = \vec{r}_0 + t\,\vec{q}$. Here $\vec{q}$ (the "blue" vector is a (displacement) vector (* it isn't (position) vector!*)

So the grammar of this (equation) is:
For a point on the line

(position) vector $=$ (starting point (position) vector) $+$ (scaling) of (displacement) vector.

In ordinary English, what the above says is just: "the position of any point of a line is (obtained by) (i) first specifying a certain position on the line, then (ii) follow a certain direction.

- (Important Remark): We get (similar) equation for a line in the space, i.e. $\mathbb{R}^3$. The only difference is that instead of 2 components, the vectors now has 3 components, i.e. instead of $x\hat{\imath} + y\hat{\jmath}$, we have $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

- Remark: Other ways of describing a line. In school math, we have learned the following way of describing a line in $\mathbb{R}^2$.

$$\text{A line in } \mathbb{R}^2 = \{(x,y)^t \colon\ y = mx + c\}$$

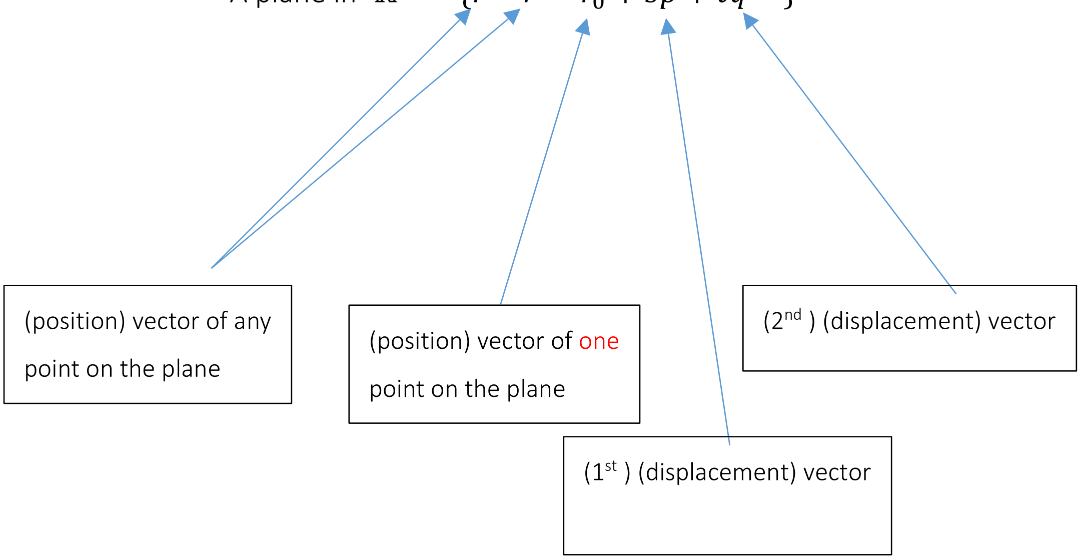
Remark: The symbol t means (x,y) should be written (vertically). The numbers m and c are two constant numbers with special (geometric) meanings.

- (Planes in $\mathbb{R}^3$) Important Point. **In principle, one can follow the method we used for describing a line in $\mathbb{R}^2$ to describe a plane in $\mathbb{R}^3$.**

The only change is (*) instead of (one) displacement vector, we have (two) displacement vectors.

The grammar is:

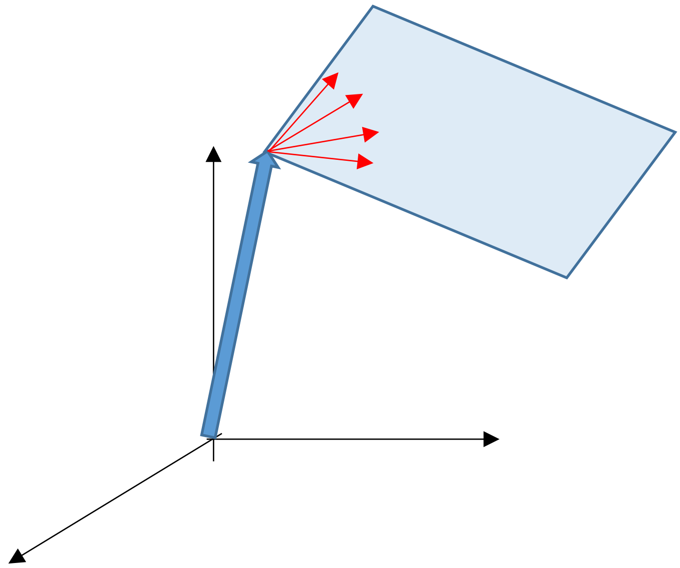
$$\text{A plane in } \mathbb{R}^3 = \{\vec{r} : \vec{r} = \vec{r}_0 + s\vec{p} + t\vec{q} \}$$



Remarks:

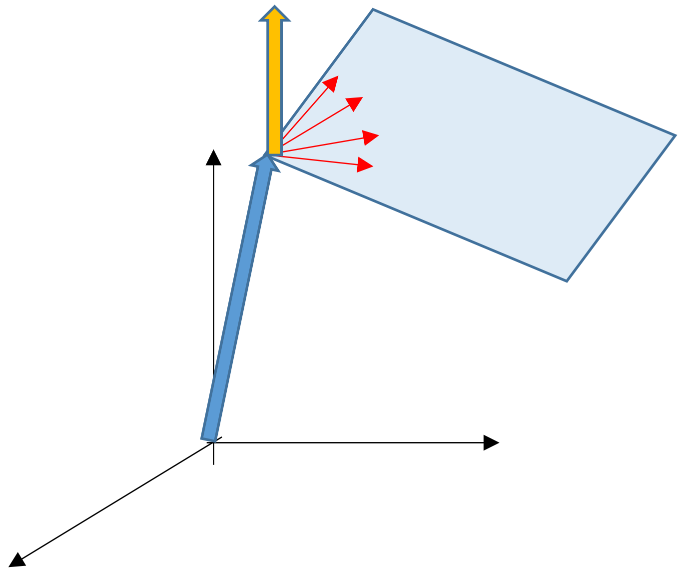
1. In the above diagram, s,t are some (scaling) factors.
2. The two (displacement) vectors have to be not on the same line (i.e. "non-collinear").
3. You can read Kai Behrend's notes, where there are more pictures on this!

- (Normal vector & Plane)
  One drawback of the above paragraph is that (there are too many choices for the two (displacement) vectors. See the picture below)!



Question: Can we get a simpler equation for a plane in $\mathbb{R}^3$?

Answer: Via normal vector. Though there are (many choices of) displacement vectors, the choices of (normal) vectors are small, i.e. either the “orange” one,



the (prolongation) of it, or (reverse) of it (i.e. opposite direction).

Notation: Let’s give the symbol $\vec{N}$ to any of such normal vector (again it’s not a (position) vector!)

Using this one can say:

A plane in $\mathbb{R}^3$ = $\{\vec{r} : (\vec{r} - \vec{r}_0) \cdot \vec{N} = 0\}$

Remark: In plain English, this sentence is saying something like: "a plane in the 3D space is the (set) of all those points $\vec{r}$ satisfying the equation $(\vec{r} - \vec{r}_0) \cdot \vec{N} = 0$."

- Other ways of talking about a plane in $\mathbb{R}^3$. (i) A plane in $\mathbb{R}^3$ = $\{(x, y, z)^t : Ax + By + Cz = 0\}$, (ii) One can also make z the (subject) of the equation and write: "A plane in $\mathbb{R}^3$ = $\{(x, y, z)^t : z = \alpha x + \beta y\}$". (Remark: You need to check that "sometimes, you cannot make z the (subject), when?).

- Concluding Remarks: In the above, we described (i) how to describe a line in the plane, (ii) how to describe a plane in the 3D space, (iii) in both cases, we have the grammatical rule: "A (line/plane) in $\mathbb{R}^2$ (respectively in $\mathbb{R}^3$) $= \{point : some\ equations\ satisfied\ by\ the\ point\}$.

- If you think about the "Concluding Remarks" carefully, you will see that one can use the same "grammatical rule" to (describe) objects such as (curve) in $\mathbb{R}^2$ (respectively in $\mathbb{R}^3$) or even (surface) in $\mathbb{R}^2$ (respectively in $\mathbb{R}^3$).
  The grammatical rule has to be of the form:

  A curve in $\mathbb{R}^2$ (respectively in $\mathbb{R}^3$) = $\{point : some\ equations\ satisfied\ by\ the\ point\}$.

- The only difference lies in the "word" equations. Let's consider an example to see what we mean here. Example: We want to describe the "circle centered at the Origin and with radius R in the plane (let's call it $K_R(O)$". This is given by

$$K_R(O) = \{(x, y)^t : some\ equations\ satisfied\ by\ x\ and\ y\}.$$

More precisely, the (equation(s)) is: $\sqrt{(x-0)^2 + (y-0)^2} = R$. Putting this back into the sentence above, we obtain

$$K_R(O) = \{(x, y)^t : \sqrt{(x-0)^2 + (y-0)^2} = R\}$$

which after some simplification, takes the form

$$K_R(O) = \{(x, y)^t : x^2 + y^2 = R^2\}$$

- Remark: The only difference between (curve/surface) and (line/plane) description is that the (equation(s)) may become (more complicated)!

- (Some Food for Thought):
  (Question 1) Write down what you think "Sphere of radius R centered at the point $(a,b,c)^t$ is,

  (Question 2) (Read the following lines first. After that you will be asked a question). **** There are at least two ways to write down (equation(s)) of a surface, i.e. (i) $z = f(x,y)$, where the left-hand side is the (subject) z and the right-hand side describe (how z is related to the variables x and y. (ii) On the other hand, one can (mix) the subject and the variables together to get one equation of the form $g(x,y,z) = 0$. Now we no longer know which one of x, y or z is the subject.***
  (Question): You are given an equation of the form $g(x,y,z) = 0$, where now $g(x,y,z) = \left(\frac{x^2}{4}\right) + \left(\frac{y^2}{3}\right) + \left(\frac{z^2}{9}\right)$. Rewrite it in the form $z = f(x,y)$ by finding out the function(s) on the right-hand side (which depends (only) on x and y).

- (A Remark & Gradient Vector) There are (names) for the two ways of describing a surface mentioned in the preceding paragraph: (i) if we write $z = f(x,y)$, we say we are having a (non-parametric) form of a surface; (ii) if we write $g(x,y,z) = 0,$ we say we are having an (implicit, i.e. "implied") form of the surface. Actually, it means we can (if we want) make z the subject, though it may be very complicated to do so.
  (Advantage of the form $g(x,y,z) = 0.$ If we describe a surface this way, then we can immediately get information about normal vectors on the surface. Let us consider one very simple example.
  Example: The surface described by $x^2 + y^2 + z^2 - 9 = 0$. I think you know that this is a sphere of radius 3 centered at the Origin. Suppose now you know the position of a point on this surface, e.g. the point $(1,0,2\sqrt{2})^t$, how can you compute the normal vector(s) to this surface at this point? The answer is given by the (Gradient Vector), which is given by

$$\left(\left.\frac{\partial g}{\partial x}\right|_{(1,0,2\sqrt{2})^t}, \left.\frac{\partial g}{\partial y}\right|_{(1,0,2\sqrt{2})^t}, \left.\frac{\partial g}{\partial z}\right|_{(1,0,2\sqrt{2})^t} \right)^t$$

  , where $\star\star|_*$ means the function $\star\star$ is (computed/evaluate) at the point $*$.
  Working this out, we see that (because $\frac{\partial g}{\partial x} = 2x, \frac{\partial g}{\partial y} = 2y,\ \frac{\partial g}{\partial z} = 2z$) a normal vector is given by $(2, 0, 4\sqrt{2}\,)^t$.

(Conclusion) The (implicit) description of surface gives a very (convenient) means to compute (normal) vectors on the surface.

Remarks

(1) The above (way of finding normal vector on (implicitly defined surfaces)) also works for curves.

(2) Since it is very inconvenient to write things like $\left.\frac{\partial g}{\partial x}\right|_{(1,0,2\sqrt{2})^t}$ or $\left.\frac{\partial g}{\partial y}\right|_{(1,0,2\sqrt{2})^t}$ we introduce the notation $g_x(1,0,2\sqrt{2})$ and $g_y(1,0,2\sqrt{2})$ to mean the same things.

Even in the case of curves in $\mathbb{R}^2$, one has the choice between (i) letting z to be subject and describe a curve by $y = f(x)$ or (ii) mixing everything and describe a curve by (implicit form) $g(x, y) = 0$.

Just as in our example to find (normal) vector(s) to the (sphere) $x^2 + y^2 + z^2 - 9 = 0$, we can find (normal) vector(s) to the circle $x^2 + y^2 - 9 = 0$ at the point $(1,2\sqrt{2})^t$ on it by (i) computing

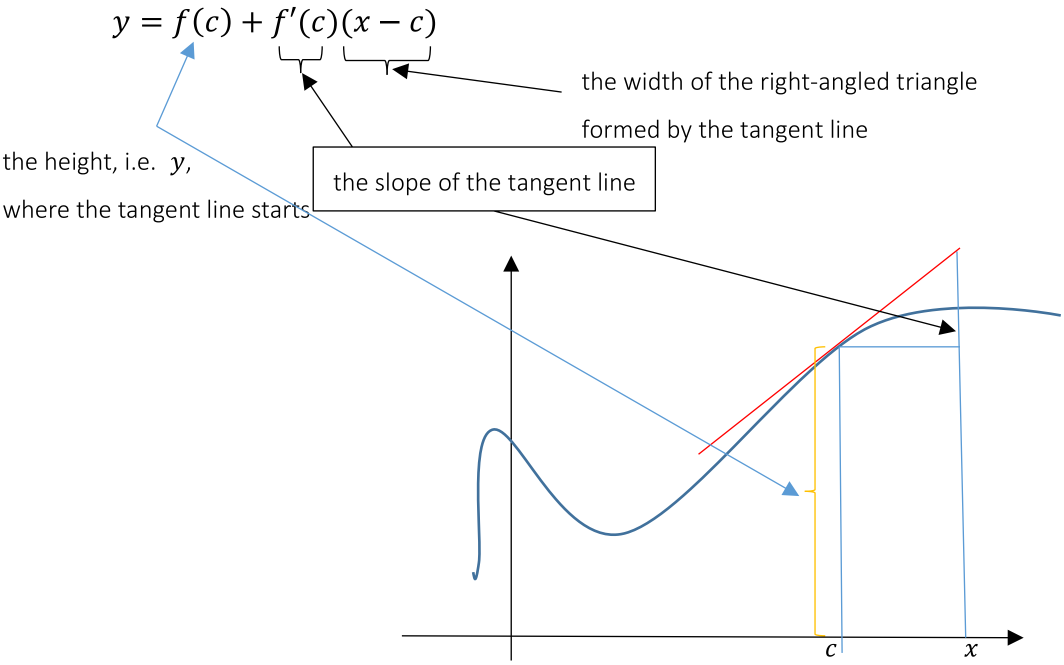
$$\left(\left.\frac{\partial g}{\partial x}\right|_{(1,2\sqrt{2})^t}, \left.\frac{\partial g}{\partial y}\right|_{(1,2\sqrt{2})^t} \right)^t$$

to obtain the normal vector to be (complete it yourself!)

- Remark: This way of computing normal using the Gradient Vector will be useful later, when we talk about (Divergence Theorem), which is a useful tool in Fluid Dynamics.

- (Equation of Tangent Line) In School Calculus, you have learned what a derivative is.
The (geometric) meaning of the (derivative) to a (curve) $y = f(x)$ at the point $x = c$ is the (slope) of the (tangent line) to this curve at $x = c$.
- (How to write down the equation of the (tangent) line mentioned above?)
Usually, in school calculus, you have to (i) use point-slope form of straight line, (ii) compute $f'(c)$, (iii) substitute into the point-slope form etc.. Cumbersome!
(Any better way to do it?) Answer is as follows:

(Equation wanted) is: $y = f(c) + f'(c)(x - c)$



- (Feature(s) of this equation) (1) y is the subject, (2) the right-hand side has only 1 variable, i.e. the variable x, (3) the (power) of this x variable is $1.$
- (Extension of this Equation) In a very very similar way, we can easily write down and (remember) the "equation of tangent plane to the surface $z = f(x, y)$ at the point $x = a, y = b$". It is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

(An Example) Let's consider the surface given by $z = \sin(xy)$. Find the equation of the tangent plane to this surface at $x = \frac{1}{4}, y = \pi.$

(Answer): $f\left(\frac{1}{4}, \pi\right) = \sin\left(\frac{\pi}{4}\right) = 1/\sqrt{2}$ Also, we have

$f_x = y\cos(xy),\ f_y = x\cos(x, y)$, so we obtain $f_x\left(\frac{1}{4}, \pi\right) = \pi\cos\left(\frac{\pi}{4}\right) = \pi/\sqrt{2}.$

Also, we have $f_y\left(\frac{1}{2}, \pi\right) = \left(\frac{1}{4}\right)\cos\left(\frac{\pi}{2}\right) = \frac{1}{4\sqrt{2}}$, so the equation of the tangent plane is $z = \frac{1}{\sqrt{2}} + \left(\frac{\pi}{\sqrt{2}}\right)\left(x - \frac{1}{4}\right) + \left(\frac{1}{4\sqrt{2}}\right)(y - \pi).$

Of course, you can further simplify the right-hand side.

- (Further Explanation of the Tangent Plane equation) Recall the equation, which is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Features: The terms $f(a,b)$,

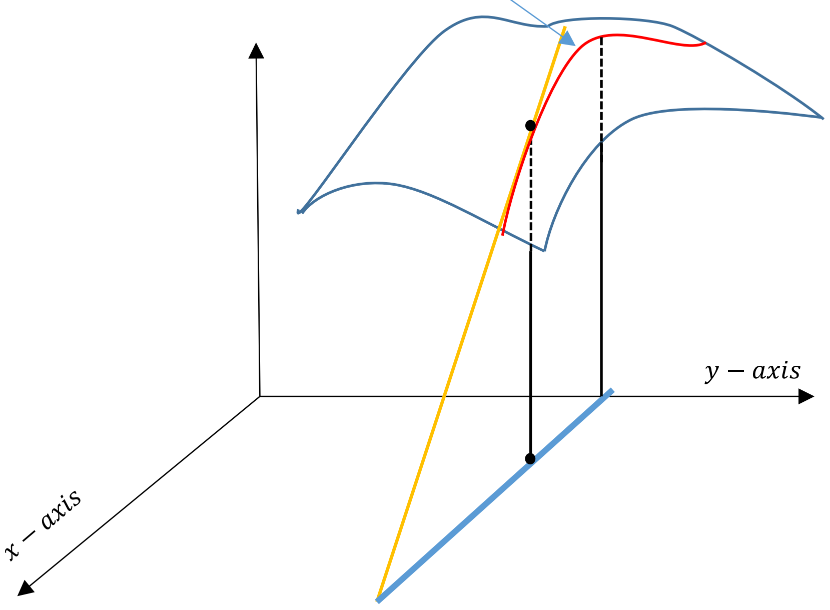
$f_x(a,b)$ and $f_y(a,b)$ are "numbers".

So, the equation is of the form $z = A + B(x-a) + C(y-b)$. Here we have used the following symbols, i.e. $A = f(a,b), B = f_x(a,b), C = f_y(a,b)$. What do you notice? (Answer): The tangent plane equation is of the form

$$z = number + number \times (x-a) + number \times (y-b)$$

This means the left-hand side is the "subject". On the right-hand side we (again) have only "numbers" and "x^1, y^1" terms.

(The terms $f_x(a,b)$ and $f_y(a,b)$). These terms means (geometrically) the (slope) of the (tangent line) to the (curve) $z = f(x,b)$ at the point $x = a, y = b$ & the slope of the curve $z = f(a,y)$ at the point $x = a, y = b$.)



- (Remarks) (1) $f_x(a,b)$ is the slope of the "orange" colored straight line.

  (2) The drawing of the (tangent line) to the (curve) $z = f(a,y)$ is omitted.

  (3) The two (non-collinear) straight line form a plane, which is the (tangent) plane we wanted.

- The definite integral $\int_a^b f(x)dx$. This (number) is the "area (under) the (curve)

$y = f(x)$, where $a \leq x \leq b$." This number can be (calculated) via (approximation) by (putting) more and more rectangles under the curve. This method is called Riemann-sum method. The number $\int_a^b f(x)dx$ is the (limiting value) of $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$, when $n \to \infty$.

- (Remarks) (1) In the above paragraph, the number ξ_i (pronounced "ksi" "ai") is any (convenient, or simple-to-calculate) point satisfying $x_{i-1} \leq \xi_i \leq x_i$.
  (2) The symbol Δx_i is the just the (width) of the subinterval $[x_{i-1}, x_i]$.
  (3) This sum and its (limiting value) are usually quite difficult to calculate, that's why in school calculus, we didn't use this method to find
$$\int_a^b f(x)dx.$$
- (Line Integral of a Scalar Field over a curve C) The idea is to do something similar to
$$\int_a^b f(x)dx$$
bearing in mind, however, that now the (domain of integration) is no longer a (straight) line interval such as $[a, b]$, but a curve C.
(How to define it?) The steps are: (1) suppose the function we want to (integrate) over the curve C is given the symbol g. Then we first (restrict) g to (sit) on the curve C. (That is, we write $g(x, f(x))$).
(2) Next, we find a (representation) of the curve. This is done in the following way. Think of the curve C as given by a (function) such as $y = f(x), a \leq x \leq b$.
(3) Then we decompose the interval $[a, b]$ by considering the subintervals $[a, x_1], [x_1, x_2], \cdots, [x_{n-1}, b]$.
(4) Each of these subintervals corresponds to part of the curve. By Pythagoras' Theorem, these pieces of the curve have lengths (approximately) equal to the numbers $\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$, where i runs from 1 to n.
(5) Introducing the (symbols) $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, we obtain $\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ which we denote by another (symbol) Δs_i. Therefore, we have $\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$.
(6) Next, we consider the sum $\sum_{i=1}^{n} g(a\ point\ on\ the\ curve) \cdot \Delta s_i$
(7) More precisely, we consider (any convenient) point $\xi_i \in [x_{i-1}, x_i]$ and the corresponding sum $\sum_{i=1}^{n} g(\xi_i, f(\xi_i)) \cdot \Delta s_i$.
(8) Now, we should notice that the expression$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ can be

simplified to the form $\Delta s_i = \sqrt{\left(\frac{\Delta x_i}{\Delta x_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i$.

(9) Now go back to (7) and let $n \to \infty$, then we see that the (limiting value) of the sum $\sum_{i=1}^{n} g(\xi_i, f(\xi_i)) \cdot \Delta s_i = \sum_{i=1}^{n} g(\xi_i, f(\xi_i)) \cdot \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i$ is the integral $\int_a^b g\big(x, f(x)\big)\sqrt{1 + (y')^2}\, dx$. This integral is called the (line integral of scalar field g over the curve C.)

- (A Special Case) If the function $g(x, y) = 1$ for each x and y, then the integral in the previous paragraph takes the form

$$\int_a^b \sqrt{1 + (y')^2} dx$$

This is just the (length) of the curve C.

- (Question:) What happens if we (represents/parametrize) the curve using a (parameter)? That is, we write each point of the curves in the form $x = x(t), y = y(t)$, where $t \in [a, b]$?

(Answer) In this case, the line integral mentioned above takes the form

$$\int_{t=a}^{t=b} g\big(x(t), y(t)\big)\sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt$$

The argument which leads to this expression is similar to the before.