Suggested Solution to Assignment 1(the first edition)

Exercise 1.1

1. Example 1,2,4,7,8 are linear, and Example 3,5,6 are nonlinear. □

2. By the definition of linearity for operators in 1.1(3), the operators in (a) and (e) are linear, others are not linear. □

3. (a) order 2 with u_{xx}, linear inhomogeneous; (b) order 2 with u_{xx}, linear homogeneous; (c) order 3 with u_{xxt}, nonlinear; (d) order 2 with with u_{tt}, u_{xx}, linear inhomogeneous; (e) order 2 with u_{xx}, linear homogeneous; (f) order 1 with u_x and u_y, nonlinear; (g) order 1 with u_x and u_y, linear homogeneous; (h) order 4 with u_{xxxx}, nonlinear. □

4. Suppose that $\mathscr{L}u_1 = g$ and $\mathscr{L}u_2 = g$ and let $u = u_1 - u_2$, then $\mathscr{L}u = \mathscr{L}u_1 - \mathscr{L}u_2 = g - g = 0$, where the operator L is linear. □

10. Since the differential equation is linear and homogeneous, its solutions form a vector space. Since $a^3 - 3a^2 + 4 = (a-2)^2(a+1)$, a basis of it is $\{e^{2x}, xe^{2x}, e^{-x}\}$. □

11. Let $u(x, y) = f(x)g(y)$, then by direct calculation we have

$$\begin{aligned} u(x,y)u_{xy}(x,y) &= f(x)g(y)f'(x)g'(y) \\ &= f'(x)g(y)f(x)g'(y) \\ &= u_x(x,y)u_y(x,y) \end{aligned}$$

Hence, $uu_{xy} = u_xu_y$ is verified. □

12. Let $u_n(x, y) = \sin nx \sinh ny$, then for $n > 0$,

$$u_{xx} + u_{yy} = -n^2 \sin(nx)\sinh(ny) + n^2 \sin(nx)\sinh(ny) = 0.$$

Thus, $u_{xx} + u_{yy} = 0$ is verified. □

Exercise 1.2

1. Using the characteristic curve method or the coordinate method, we have $u(t, x) = f(3t - 2x)$. Setting $t = 0$ yields the equation $f(-2x) = \sin x$. Letting $w = -2x$ yields $f(w) = -\sin(w/2)$. Therefore, $u(t, x) = \sin(x - 3t/2)$. □

2. Let $v = u_y$, then $3v + v_x = 0$. Thus we have $v(x, y) = f(y)e^{-3x}$, i.e., $u_y(x, y) = f(y)e^{-3x}$, which implies $u(x, y) = F(y)e^{-3x} + g(x)$, where both F and g are arbitrary (differentiable) functions. □

3. The characteristic curves satisfy the ODE: $dy/dx = 1/(1 + x^2)$, which implies $y = \arctan x + C$. Thus $u(x, y) = f(y - \arctan x)$. We omit the easy figure here. □

5. (in second edition)When $x \neq 0$, the characteristic curves satisfy the ODE: $dy/dx = y/x$, which implies $y = C_1x$ and then general $u(x, y) = f(y/x)$; When $y \neq 0$, we have that the characteristic curves are $x = C_2y$ and general solution is $u(x, y) = g(x/y)$. We claim that the general solution on the whole space has the form $u(x, y) = h(arctan(y/x))$, where h is an arbitrary (differentiable) function and satisfies $h(-\pi/2) = h(\pi/2)$, $h'(-\pi/2) = h'(\pi/2)$. Indeed, let $f(x) = h(arctan(x))$,$g(x) = h(arctan(1/x))$ and we find f, g are differentiable and thus $h(arctan(y/x))$ solves the equation. Whats more, since $y = arctan(x) \in C^1(-\infty, \infty)$, is one to one and onto $[-\pi/2, \pi/2]$, so when $x \neq 0$, the solution has the form $u(x, y) = h(arctan(y/x))$; letting the solution exists on the whole space,$h(-\pi/2) = h(\pi/2)$,$h'(-\pi/2) = h'(\pi/2)$ is easily attained by fixing y and moving x to 0.

5. (6. in second edition) The characteristic curves satisfy the ODE: $dy/dx = 1/\sqrt{1-x^2}$, which implies $y = \arcsin x + C$. Thus $u(x,y) = f(y - \arcsin x)$. Setting $x = 0$ yields the equation $f(y) = y$, and then $u(x,y) = y - \arcsin x$. □

6. (7. in second edition) (a)The characteristic curves satisfy the ODE: $dy/dx = x/y$, which implies $y^2 = x^2 + C$ and then $u(x,y) = f(y^2 - x^2)$. Setting $x = 0$ yields the equation $f(y^2) = e^{-y^2}$. Letting $w = y^2$ yields $f(w) = e^{-w}$ and $u(x,y) = e^{x^2-y^2}$. (b)The characteristic curves must go through $(0,y)$, which implies $C = y^2 - x^2 = y^2 \geq 0$. Thus, the solution is uniquely determined in $\{(x,y)|y^2 - x^2 \geq 0\}$. □

7. (8. in second edition) Change variables to $x = ax + by, y = bx - ay$. By the chain rule,

$$u_x = au_{x'} + bu_{y'}, u_y = bu_{x'} - au_{y'}$$

. We have $(a^2+b^2)u_{x'} + cu = 0$ which implies $u(x,y) = f(y)e^{-cx/(a^2+b^2)}$ and then $u(x,y) = f(bx - ay)e^{-c(ax+by)/(a^2+b^2)}$, where f is a arbitrary (differentiable) function. □

8. (10. in second edition)Note that $u(x,y) = e^{x+2y}/4$ is a special solution of the inhomogeneous equation, and by the result of Exercise 1.2.7 above, the general solution of the corresponding homogeneous equation is $f(x-y)e^{-(x+y)/2}$. Thus the general solution of $u_x + u_y + u = e^{x+2y}$ is

$$u(x,y) = f(x-y)e^{-(x+y)/2} + e^{x+2y}/4,$$

where f is a arbitrary function. Let $y = 0$, and then we have $f(x)e^{-x/2} + e^{x/4} = 0$, i.e. $f(x) = -e^{3x/2}/4$. So the solution is $u(x,y) = (e^{x+2y} - e^{x-2y})/4$.

9. (11. in second edition) By changing variables,$x' = ax+by, y' = bx-ay$. The original equation is equivalent to the following form

$$(a^2+b^2)u_{x'} = f(\frac{ax'+by'}{a^2+b^2}, \frac{bx'-ay'}{a^2+b^2}).$$

Therefore, we have the general solution to the above equation is

$$u(x',y') = \frac{1}{a^2+b^2}\int_0^{x'} f(\frac{as'+by'}{a^2+b^2}, \frac{bs'-ay'}{a^2+b^2})\, ds' + g(y'),$$

where we let $g(y') = u(0,y')$, and g is a arbitrary function. Returning back to the original parameters, the integral changes to be the integral along the line

$$L = \{(m,n); 0 \leq s' = am + bn \leq ax + by, y' = bm - an = bx - ay\}.$$

When denoting s the parameter of arc length, we have $ds = \sqrt{(dm)^2 + (dn)^2}$. Note that along the line L the condition $bm - an = bx - ay$ is satisfied. Thus, $b(dm) - a(dn) = 0$, and then $ds = \frac{\sqrt{a^2+b^2}}{a}dm$, $ds' = a(dm) + b(dn) = \frac{a^2+b^2}{a}$. Hence, the solution is

$$u(x,y) = \frac{1}{(a^2+b^2)^{1/2}}\int_L f\, ds + g(bx - ay),$$

where L is shown above(actually, the line segment is not from the y axis). □

10. (12. in second edition) the directions of the new coordinate axes x' and y' are (a,b) and $(b,-a)$ respectively in the old coordinate. And $(a,b)\cdot(b,-a) = 0$ implies that the the new coordinate axes are orthogonal.

11. (13. in second edition) Using Coordinate Method, we change variables $x' = x + 2y, y' = 2x - y$, then the original equation is changed into

$$5u_{x'} + y'u = x'y'.$$

Note that $u(x', y') = x' - \frac{5}{y'}$ is a special solution and $u(x', y') = f(y')e^{-(x'y'/5)}$ is the general solution of the corresponding homogeneous equation. Hence the general solution of original equation is

$$u(x, y) = f(2x - y)e^{-\frac{(x+2y)(2x-y)}{5}} + x + 2y - \frac{5}{2x - y}$$

where f is an arbitrary (differentiable) function. $\square$

Exercise 1.3

1. According to Example 2, we only need to add the resistance in the transverse equation. Under the assumption that the resistance is proportional to the velocity, the transverse equation becomes

$$\left.\frac{Tu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} + \int_{x_0}^{x_1} -ku_t dx = \int_{x_0}^{x_1} \rho u_{tt} dx$$

where $k > 0$ is a coefficient depending on the property of the medium (e.g. the density of the medium). Note that the direction of resistance should be opposite to the velocity, thus we have the negative sign before k. The equation, differentiated, says that

$$(Tu_x)_x - ku_t = \rho u_{tt}$$

That is,

$$u_{tt} - c^2 u_{xx} + ru_t = 0$$

where $c = \sqrt{T/\rho}$,$r = k/\rho > 0$.

2. By the assumption in the question, $T = -\int_l^x \rho g dx = \rho g(l - x)$. Then, similar to Example 2 given in the text, we can obtain the PDE satisfied by the chain,

$$((l - x)\rho g u_x)_x = \rho u_{tt}.$$

3. The heat energy contained in the part between x_0 and x_1 of the thin rod at time t is $Q(t) = \int_{x_0}^{x_1} c\rho uAdx$ where A is the area of of the cross section. The heat energy flowing across the two ends per unit time is $ku_xA|_{x_0}^{x_1}$ and that flowing out the lateral sides per unit time (by Newton's law of cooling) is $\int_{x_0}^{x_1} \mu P(u - T_0)dx$ where P is the perimeter of the cross section and μ is the conductance across the contact surface. Hence,

$$\frac{dQ(t)}{dt} = ku_xA|_{x_0}^{x_1} - \int_{x_0}^{x_1} \mu P(u - T_0)dx,$$

which implies

$$Ac\rho u_t = A(ku_x)_x - \mu P(u - T_0)$$

5. Let $u(x, t)$ be the concentration (mass per unit length) of the dye at position x of the pipe at time t. The mass of dye is $M(t) = \int_{x_0}^{x} u(y, t)dy$, so $\frac{\partial M}{\partial t} = \int_{x_0}^{x} u_t(y, t)dy$. Then by the Fick's law,

$$\frac{\partial M}{\partial t} = \text{flow in} - \text{flow out} = V(u(x_0, t) - u(x, t)) + ku_x(x, t) - ku_x(x_0, t).$$

Differentiating with respect to x, we get $u_t = ku_{xx} - Vu_x$. $\square$

6. Since the heat flow depends only on t and on the distance $r = \sqrt{x^2 + y^2}$ to the axis of the cylinder,

$$u(x, y, z, t) = u(\sqrt{x^2 + y^2}, t) = u(r, t).$$

Then by the chain rule,

$$u_x = u_x x/r, \qquad u_y = u_r y/r, \qquad u_z = 0,$$
$$u_{xx} = u_{rr}x^2/r^2 + u_r(r^2 - x^2)/r^3, \qquad u_{yy} = u_{rr}y^2/r^2 + u_r(r^2 - y^2)/r^3, \qquad u_{zz} = 0.$$

Therefore, $u_t = k(u_{xx} + u_{yy} + u_{zz}) = k(u_{rr} + u_r/r)$. □

7. Since the heat flow depends only on t and on the distance $r = \sqrt{x^2 + y^2 + z^2}$ to the cylinder,

$$u(x, y, z, t) = u(\sqrt{x^2 + y^2 + z^2}, t) = u(r, t).$$

Then by the chain rule,

$$u_x = u_x x/r, \qquad u_y = u_r y/r, \qquad u_z = u_r z/r,$$
$$u_{xx} = \frac{u_{rr}x^2}{r^2} + \frac{u_r(r^2 - x^2)}{r^3}, \qquad u_{yy} = \frac{u_{rr}y^2}{r^2} + \frac{u_r(r^2 - y^2)}{r^3}, \qquad u_{zz} = \frac{u_{rr}z^2}{r^2} + \frac{u_r(r^2 - z^2)}{r^3}.$$

Therefore, $u_t = k(u_{xx} + u_{yy} + u_{zz}) = k(u_{rr} + 2u_r/r)$. □

Exercise 1.4

1. Setting $u(x,t) = f(t) + x^2$ yields the equations $f'(t) = 2$ and $f(0) = 0$. Hence, $f(t) = 2t$ and $u(x,t) = 2t + x^2$ is a solution of the diffusion equation. □

2. (a) No heat flows across the boundary, by the Fouriers law, we have $\partial u/\partial x = 0$;
   (b) No gas flows across the boundary, by the Ficks law, we have $\partial u/\partial x = 0$;
   (c) No heat or gas flows across the boundary, by the Fouriers or Ficks law, we have $\partial u/\partial n = 0$. □

3. After long time, if this homogeneous body reaches a steady state, then $\partial_t u = 0$, therefore, $u_{xx} = 0$. Since it is insulated, therefore, we have $u \equiv constant$. So the steady -state temperature is $\frac{\int_D f\,d\mathbf{x}}{\int_D d\mathbf{x}}$. □

4. (in second edition) (a)the steady-state temperature $u(x)$ satisfies $u_{xx} = -f(x)$ for $0 < x < l$ and $u(0) = u(l) = 0$. Hence, by integrating, $u_x = C$ for $0 < x < l/2$ and $= C - H(x - l/2)$ for $l/2 < x < l$, where C is a constant to be determined later. By integrating and using $u(0) = 0$, we obtain $u(x) = Cx$ for $0 < x < l/2$ and $= Cx - H(x - l/2)^2/2$ for $l/2 < x < l$. Using $u(l) = 0$, we get $C = Hl/8$. Thus, $u(x) = Hlx/8$ for $0 < x < l/2$ and $= Hlx/8 - H(x - l/2)^2/2$ for $l/2 < x < l$. (b)$u_x > 0$ if and only if $0 < x < 5l/8$, $u_x < 0$ if and only if $5l/8 < x < l$. That is, the hottest point is $x = 5l/8$ and the temperature there is $u(5l/8) = 9Hl^2/128$.

4. (5. in second edition)

   $ku_z + Vu = 0$ on $z = a$

Exercise 1.5

1. The general solution of the ODE: $\frac{d^2u}{dx^2} + u = 0$ is $u(x) = C_1 \cos x + C_2 \sin x$. Hence, to satisfy the boundary conditions,

$$C_1 = 0 \quad \text{and} \quad C_1 \cos(L) + C_2 \sin(L) = 0.$$

   Therefore, $C_1 = 0$ and $C_2 \sin(L) = 0$. So the solution $u \equiv 0$ if and only if L is not an integer multiple of π. □

2. (a) The solution is not unique. Indeed, if there exists a solution u_0, then $u_0 + C(e^{-x} - 2)$ is also a solution of equation for any constant C.

(b) The solution does not necessarily exist, since the condition that $f(x)$ must satisfy for the existence is:

$$\int_0^l f(x)dx = \int_0^l [u''(x) + u'(x)]\ dx = [u'(l) + u(l)] - [u'(0) + u(0)] = 0. \qquad \square$$

3. The general solution of $u''(x) = 0$ is $u(x) = ax + b$, where a and b are constants. Hence, when we do the $+$ case, a and b have to satisfy $a + kb = 0$ and $a + k(a + b) = 0$, and then solution(s) of the boundary problem would be

$$u(x) = \begin{cases} 0 & \text{if } k \neq 0 \\ b & \text{if } k = 0 \end{cases};$$

when we do the $-$ case, the solution(s) of the boundary problem would be

$$u(x) = \begin{cases} 0 & \text{if } k \neq 0, 2 \\ b & \text{if } k = 0 \\ -2bx + b & \text{if } k = 2 \end{cases}.$$

If $k = 2$, the boundary problem is unique for the $+$ case, but not for the $-$ case. $\square$

4. (a) Adding a constant C to a solution will give another solution, so we do not have uniqueness if there is a solution;

(b) Integrating $f(x, y, z)$ on D and using the divergence theorem, we obtain

$$\iiint_D f(x, y, z)dxdydz = \iiint_D \Delta u dxdydz = \iiint_D \nabla \cdot \nabla u dxdydz = \iint_{\partial D} \nabla u \cdot n dS = 0$$

(c) For heat flow or diffusion, u is a physical quantity in terms of time t. The equation here can only describe the derivatives of u with respect to (x, y, z). So (a) shows that u up to a constant has the same derivatives with respect to (x, y, z).
Since for heat flow and diffusion $u_t = k\Delta u = kf(x, y, z)$, (b) shows that to satisfy the boundary condition(insulated solid or impermeable container), the change of the whole heat energy or the whole substance with respect to time, which is proportion to $\iiint_D \frac{\partial u}{\partial t} dxdydz = k \iiint_D f(x, y, z)dxdydz$, has to be 0. $\square$

5. (a) The characteristic curves satisfy the ODE:

$$\frac{dy}{dx} = y.$$

By separation of the variables and integration, we see that the ODE has the solutions

$$y = Ce^x.$$

Any solution $u(x, y)$ stays constant along the characteristic curves. It follows that $u(x, y) = f(e^{-x}y)$ is the general solution of this PDE. Applying the boundary condition $u(x, 0) = x$, we get

$$u(x, 0) = f(0) = x.$$

But $f(0)$ is a constant, and so the equality can not hold for all x. There is no solution to this boundary value problem.

(b) Applying the boundary condition $u(x, 0) = 0$, we would have

$$u(x, 0) = f(0) = 0.$$

Since there are infinitely many smooth function $f(x)$ with $f(0) = 0$, (for examples, $f(x) = 1 + cx$, where c is an arbitrary real number), setting $u(x, y) = f(e^{-x}y)$ we have infinitely many solutions to the BVP. □

6. Check Example 3 in Section 1.2 of the text book. □

Exercise 1.6

1. Indeed, we check the sign of the "discriminant" $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$.

   (a) $\mathscr{D} = [(-1-3)/2]^2 - 1 \cdot 1 = 3 > 0$, so it is hyperbolic.

   (b) $\mathscr{D} = [6/2]^2 - 9 \cdot 1 = 0$, so it is parabolic. □

2. Indeed, its discriminant is

$$\mathscr{D} = (xy)^2 - (1+x)(-y^2) = (x^2 + x + 1)y^2 = [(x + 1/2)^2 + 3/4]y^2,$$

   So it is hyperbolic in $\{y \neq 0\}$, parabolic on $\{y = 0\}$, and elliptic nowhere. We omit the easy figure here. □

3. In the equations of the form (1), suppose $A = (a_{ij})$, $n = (a_i)$ and $b = a_0$. Denote $B = (b_{ij})$ as the matrix of the rotation. Therefore, the new variables (ξ, η) satisfy $(\xi, \eta)^T = B(x, y)^T$, and the new coefficients satisfy $A' = BAB^T$, $n' = nB^T$ and $b' = b$. So the rotationally invariant equations have to satisfy

$$A = BAB^T, \quad n' = nB^T \quad \forall \text{ normal matrix } B.$$

   Thus, A is a unit matrix multiple of a constant a, and $n = 0$. So all rotationally invariant equations of the form (1) have the form $a(u_{xx} + u_{yy}) + bu = 0$. □

4. It is parabolic since its discriminant $\mathscr{D} = (-4/2)^2 - 1 \cdot 4 = 0$. By direct subtitution, if $u(x+y) = f(y+2x) + xg(y+2x)$, then $u_{xx} = 4f''(y+2x) + 4xg''(y+2x) + 4g'(y+2x)$, $u_{xy} = 2f''(y+2x) + 2xg''(y+2x) + g'(y+2x)$ and $u_{yy} = f''(y+2x) + xg''(y+2x)$, and then the equation is satisfied. □

5. Let $u = ve^{(\alpha x + \beta y)}$, then

$$u_x = (v_x + \alpha v)e^{(\alpha x+\beta y)} \qquad u_y = (v_y + \beta v)e^{(\alpha x+\beta y)}$$
$$u_{xx} = (v_{xx} + 2\alpha v_x + \alpha^2 v)e^{(\alpha x+\beta y)} \qquad u_{yy} = (v_{yy} + 2\beta v_y + \beta^2 v)e^{(\alpha x+\beta y)}$$

   Hence, by direct substituting,

$$(v_{xx} + 2\alpha v_x + \alpha^2 v) + 3(v_{yy} + 2\beta v_y + \beta^2 v) - 2(v_x + \alpha v) + 24(v_y + \beta v) + 5v = 0,$$

$$v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v = 0.$$

   Let $\alpha = 1$ and $\beta = -4$, the equation turns out to be $v_{xx} + 3v_{yy} - 44v = 0$. By setting $x' = x$ and $y' = \sqrt{3}y$, the equation turns out to be $v_{x'x'} + v_{y'y'} - 44v = 0$. □

6. (a) It is hyperbolic since its discriminant $\mathscr{D} = (1/2)^2 > 0$;

   (b) Set $v = u_y$, we have $3v + v_x = 0$ which implies $v(x, y) = f(y)e^{-3x}$ and thus $u(x, y) = F(y)e^{-3x} + g(x)$, where F, g are arbitrary (differential) functions.

(c) Setting $y = 0$ yields

$$e^{-3x} = u(x, 0) = F(0)e^{-3x} + g(x)$$
$$0 = u_y(x, 0) = F'(0)e^{-3x}.$$

Therefore,

$$u(x, y) = (F(y) + 1 - F(0))e^{-3x},$$

where $F(y)$ satisfy $F'(0) = 0$. By setting $F(y) = ny^2, n = 1, 2, \cdots$, we obtain infinitely many solutions $u(x, y) = (ny^2 + 1)e^{-3x}$ of the problem. □