Tutorial 5 for MATH4220

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1. Discuss why there is no maximum principle for wave equation?

   Consider the following Cauchy problem:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & -\infty < x < +\infty, \quad t > 0 \\ u(x, t=0) = 0, & \partial_t u(x, t=0) = \sin x, \quad -\infty < x < +\infty \end{cases}$$

   And the unique solution is given by d'Alembert formula:

$$u(x,t) = \frac{1}{2}\{\cos(x+t) - \cos(x-t)\} = -\sin x \sin t, \quad -\infty < x < \infty, t > 0$$

   Then $u(x,t)$ attains its maximum 1 only at the interior points $(\frac{\pi}{2} \pm 2n\pi, \frac{3\pi}{2} + 2m\pi)$ or $(\frac{3\pi}{2} \pm 2n\pi, \frac{\pi}{2} + 2m\pi)$ for $n, m = 0, 1, 2, \cdots$. However, $u(x,t) = 0$ on the boundary $\{(x,t) : t = 0\}$. Therefore there is no maximum principle for the Cauchy problem for the 1-dimensitonal wave equation.

   Remark: The key is to find an counterexample.

2. Use the Green's function of the heat equation to show that the backward heat equation is not well-posed.

   Note that $S(x,t)$ satisfies $u_t = ku_{xx}$ for any $t > 0$, and $S(0,t) \to \infty$ as $t \to 0^+$. Then $u(x,t) = S(x, t+1)$ solves $u_t = ku_{xx}$ for $t > -1$. Then $S(0,t) \to \infty$ as $t \to -1^+$, which implies that there is no solution for the backward heat equation with initial data $u(x,0) = S(x,1) = \frac{1}{\sqrt{4k\pi}} e^{-\frac{x}{4k}}$, hence the backward heat equation is not well-posed.

3. Using **Reflection Method** to solve the following problem

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0, & x > 0, t > 0 \\ u(x, t=0) = \phi(x), \partial_t u(x, t=0) = \psi(x), & x > 0 \\ \partial_x u(x=0, t) = 0, & t > 0 \end{cases}$$

   Solution: Use the reflection method, and first consider the following Cauchy Problem:

$$\begin{cases} \partial_t^2 v - c^2 \partial_x^2 v = 0, & -\infty < x < \infty, t > 0 \\ v(x, t=0) = \phi_{even}(x), \partial_t v(x, t=0) = \psi_{even}(x), & -\infty < x < \infty \end{cases}$$

*Any questions on notes, please contact me at rzhang@math.cuhk.edu.hk where $\phi_{even}(x)$ and $\psi_{even}(x)$ are even extension of ϕ and ψ. Then the unique solution is given by d'Alembert formula:

$$v(x,t) = \frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)dy$$

And since $\phi_{even}(x)$ and $\psi_{even}(x)$ are even, so is $v(x,t)$ for $t > 0$, which implies

$$\partial_x v(x=0,t) = 0, t > 0.$$

Set $u(x,t) = v(x,t), x > 0$, then $u(x,t)$ is the unique solution of Neumann Problem on the half-line. More presicely, if $x > ct$,

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

if $0 < x < ct$,

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(ct-x)] + \frac{1}{2c}\Big\{\int_0^{ct-x} \psi(y)dy + \int_0^{x+ct} \psi(y)dy\Big\}.$$