Tutorial 11 for MATH4220

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1. Δ_n is invariant under rotation.

   Any rotation in three dimensions is given by

   $$x' = Ox$$

   where $O = (o_{ij})$ is an othogonal matrix, that is, $O^t O = OO^t = I$. Therefore,

   $$\Delta u = \sum_{i,j=1}^{n} \delta_{ij} u_{ij} = \sum_{i,j=1}^{n} \delta_{ij} \partial_i \Big(\sum_{k=1}^{n} u_{x'_k} \frac{dx'_k}{dx_i}\Big) = \sum_{i,j=1}^{n} \delta_{ij} \partial_i \Big(\sum_{k=1}^{n} u_{x'_k} o_{ki}\Big) = \sum_{i,j=1}^{n} \delta_{ij} \sum_{k,l=1}^{n} u_{x'_k x'_l} o_{ki} o_{lj}$$
   $$= \sum_{i,k,l=1}^{n} u_{x'_k x'_l} o_{ki} o_{li} = \sum_{k,l=1}^{n} u_{x'_k x'_l} \delta_{kl} = \Delta' u$$

   where we have used $\sum_{i=1}^{3} o_{ki} o_{li} = \delta_{kl}$.

2. For the three-dimensional laplacian

   $$\Delta_3 = \partial_x^2 + \partial_y^2 + \partial_z^2$$

   it is natural to use spherical coordinates (r, θ, ϕ). First, consider the chain of variables $(x, y, z) \to (s, \phi, z)$ which is given by

   $$x = s\cos\phi$$
   $$y = s\sin\phi$$
   $$z = z$$

   By the two-dimensional Laplace calculation, we have

   $$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s} u_s + \frac{1}{s^2} u_{\phi\phi}.$$

   Second, consider the chain of variables $(s, \phi, z) \to (r, \phi, \theta)$ which is given by

   $$s = r\sin\theta$$
   $$z = r\cos\theta$$
   $$\phi = \phi$$

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By the two-dimensional Laplace calculation, we have

$$u_{ss} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Thus we have

$$\Delta_3 u = u_{xx} + u_{yy} + u_{zz} = \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi} + u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

And note that $s = r\sin\theta$ and $u_s = u_r\frac{\partial r}{\partial s} + u_\theta\frac{\partial \theta}{\partial s} = u_r\frac{s}{r} + u_\theta\frac{\cos\theta}{r}$. Therefore

$$\Delta_3 u = \frac{1}{r^2}\cot\theta u_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} + u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

3. Question2 of quiz2.