Section 8. Some Topics in Multidimensional Conservation Laws

§8.1 Introduction

$$\begin{array}{l} \partial_t u + \text{div } F(u) = S(u,x,t) \\ t \in \mathbb{R}_1^+,\ x \in \Omega \subset \mathbb{R}^m,\ u \in \mathbb{R}^n,\ F = (F,(u),\cdots,F_m(u)),\ F_i(u) \in \mathbb{R}^n \end{array} \tag{8.1}$$

(8.1) is a system of first order quasilinear equations. It is called a system of balance laws.

u: density vector, $F(u)$: flux vector, $S(u;x,t)$: external forcing. In the case without external forces,

$$\begin{aligned} \partial_t u + \nabla \cdot F(u) &= 0 \\ \int_\Omega u(x,t)dx &= \text{const.} \end{aligned}$$

which is called a system of conservation laws.

Example: Compressible Euler System

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{conservation of mass} \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + pI) = 0 & \text{conservation of momentum} \\ \partial_t(\rho E) + \operatorname{div}(\rho u E + pu) = 0 & \text{conservation of energy} \end{cases} \quad (8.2)$$

$\rho(x,t)$: density, $u(x,t)$: velocity vector, p: pressure, E: total energy, $E = e + \frac{1}{2}|u|^2$, e: internal energy, $\frac{|u|^2}{2}$: kinetic energy
Equation of states: T: temperature, S: entropy

$$TdS = de - p/\rho^2 d\rho$$

In particular, for ideal polytropic fluid

$$e(\rho, p) = \frac{p}{\rho(\nu - 1)} = \frac{T}{\gamma - 1} \qquad e^s = p\rho^{-\gamma}$$

Definition 8.1 Set $A_j(u) = \nabla F_j(u)$, $n \times n$ matrix, and let $w \in \mathbb{R}^n \setminus \{0\}$ be any given direction. (8.1) is said to be hyperbolic in the direction w, if

$$\sum_{j=1}^{n} w_j A_j(u)$$

has n real eigenvalues

$$\lambda_1(w,u) \leq \lambda_2(w,u) \leq \cdots \leq \lambda_n(w,u)$$

with a complete right eigenvectors

$$r_1(w,u), \quad r_2(w,u), \cdots, r_n(w,u)$$

If (8.1) is hyperbolic in all directions, then (8.1) is said to be hyperbolic.

Example: The compressible Euler system (8.2) is always hyperbolic $\forall w \in \mathbb{R}^n \setminus \{0\}$.

Sound wave family $\lambda_{\pm}(u, w) = u \cdot w \pm c|w|$, where $c = \sqrt{\gamma(\frac{p}{\rho})}$: sound speed.

Entropy wave family $\lambda_0(u, w) = u \cdot w$

(Vorticity wave family)

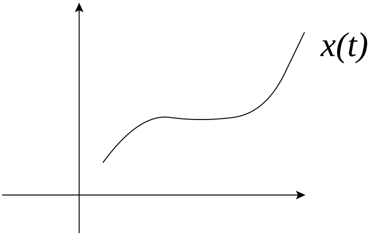
Definition 8.2 A bounded, measurable function u is called a weak solution of (8.1) iff

$$\iint \{\phi_t u + \nabla\phi \cdot F(u)\phi s\} dxdt = 0 \qquad \forall \phi \in c_0^\infty$$

in 1-D without external force:

$$\int_{\mathbb{R}}\int_0^t (\partial_t \phi u + \partial_x \phi F(u)) dxdt = 0$$

$$\begin{aligned} [u] &= u(x(t)+, t) - u(x(t)-, t) \\ [F(u)] &= F(u(x(t)+, t)) - F(u(x(t)-, t)) \end{aligned}$$



Then

$$\dot{x}(t)[u] = [F(u)]$$

Rankine-Hugeniet condition

§8.2 Friedrichs Theory for Symmetric Hyperbolic System

Consider

$$\partial_t u + \sum_{j=1}^{m} A_j \, \partial_{x_j} u = 0, \qquad t > 0, \quad x \in \mathbb{R}^m \tag{8.3}$$

$u \in \mathbb{R}^n$, A_j : $n \times n$ smooth matrix.

Definition 8.3 System (8.3) is said to be symmetrizable, if $\exists$ smooth positive definite matrix $\tilde{A}_0$, such that

(1) $\tilde{A}_0 > 0, \qquad \tilde{A}_0^* = \tilde{A}_0$

(2) $\tilde{A}_j = \tilde{A}_0 A_j$ is symmetric, i.e. $\tilde{A}_j^* = \tilde{A}_j, j = 1, \cdots m$

(3) $\tilde{A}_0 \, \partial_t u + \sum_{j=1}^m \tilde{A}_j \, \partial_{x_j} u = 0$

Remark 8.1 If a system is symmetrizable, then it must be hyperbolic, i.e. for any $w \in \mathbb{R}^m \setminus \{0\}$, $A = A(w) = \sum_{j=1}^m w_j \, A_j$ has n real eigenvalues

$$\lambda_1(w) \le \lambda_2(w) \le \cdots \lambda_n(w)$$

with a full set of right eigenvector

$$\begin{array}{l} r_1(w), r_2(w), \cdots, r_n(w) \\ A(w)\nu_i(w) = \lambda_i(w) r_i(w), \quad i = 1, \cdots, n \end{array}$$

Let the corresponding left eigenvector $l_k(w)$ be normalized so that

$$l_k^*(w)A(w) = \lambda_k(w) l_k^t(w), \qquad l_k^*(w)\nu_j(w) = \delta_{kj}$$

Example: Consider the 3-D compressible Euler System

$$\begin{cases} D_t\rho + \rho \text{ div } u = 0 \\ \rho D_t u + \rho\nabla T + T\nabla\rho = 0 \\ D_t T + (\nu - 1)T \text{ div } u = 0 \end{cases}$$

$D_t = \partial_t + u \cdot \nabla$ material derivate.

If we linearize the system around any non-vacuum state, e.g. $(\rho_0, 0, T_0)$, then the linearized system is symmetrizable.

$$\tilde{A}_0(\rho_0, 0, T_0) = \begin{pmatrix} \rho_0^{-1}T_0 & 0 & 0 \\ 0 & \rho_0 I_3 & 0 \\ 0 & 0 & \dfrac{\rho_0 T_0^{-1}}{\gamma - 1} \end{pmatrix}$$

Energy Principle: Consider the Cauthy problem

$$\begin{cases} \sum_{j=0}^{m} \tilde{A}_j \, \partial_{x_j} u + B(x,t)u = F, \quad x_0 = t \\ u(x_0 = 0, x_1, \cdots, x_m) = u_0(x_1, \cdots, x_m) = u_0(x) \end{cases} \tag{8.4}$$

Assumptions:

(1) $A = (\tilde{A}_0, \tilde{A}_1, \cdots, \tilde{A}_m)$ and B are smooth, F is also smooth.

(2) $\tilde{A}_j$ is symmetric, and $\tilde{A}_0$ is positive definite.

$$E(t) = (\tilde{A}_0 u, u)$$

$$(w, \upsilon) = \int_\Omega w(x) \cdot \upsilon(x) dx = \sum_{j=1}^{n} \int_\Omega w_j(x) \upsilon_j(x) dx$$

$$||w||_0 = (w, w)^{\frac{1}{2}}$$

Theorem 8.1 $\exists$ uniform constant $c = c(\tilde{A}_0) > 0$, such that for any smooth solution $u(x,t)$, the following stability estimate holds

$$\max_{0\le t\le T} ||u(t)||_0 \le c^{-1} \exp\left(\frac{1}{2}c^{-1}\left|\text{div } \tilde{A} + B + B^*\right|_{L^\infty} T\right)\left(||u_0||_0 + \int_0^T ||F(t)||_0 dt\right) \quad (8.5)$$

here div $\tilde{A} = \partial_t \tilde{A}_0 + \sum_{j=1}^m \partial_{x_j} A_j$.

Remark 8.2 This simple energy principle guarantees the well-posedness theory for such a linear system (Friedrichs).

Proof of (8.3)

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{d}{dt}(\tilde{A}_0 u, u) = (\tilde{A}_0 u, \partial_t u) + (\tilde{A}_0\, \partial_t u, u) + (\partial_t \tilde{A}_0 u, u)\\ &= 2(\tilde{A}_0\, \partial_t u, u) + (\partial_t \tilde{A}_0 u, u)\\ &= 2(u, \tilde{A}_0\, \partial_t u) + (\partial_t \tilde{A}_0 u, u)\\ &= -2\left(u, \sum_{j=1}^{m} \tilde{A}_j\, \partial_{x_j} u\right) - 2(u, Bu) + 2(u, F) + (\partial_t \tilde{A}_0 u, u)\end{aligned}$$

$$\begin{aligned}\partial_{x_j} < u, \tilde{A}_j u > &= < \partial_{x_j} u, \tilde{A}_j u > + < u, \tilde{A}_j\, \partial_{x_j} u > + < u, \partial_{x_j} \tilde{A}_j u >\\ &= 2 < u, \tilde{A}_j\, \partial_{x_j} u > + < u, \partial_{x_j} \tilde{A}_j u >\end{aligned}$$

so,

$$2(u, \tilde{A}_j \, \partial_{x_j} u) = -(u, \partial_{x_j} \tilde{A}_j u)$$

Thus

$$\begin{aligned} \frac{d}{dt} E(t) &= (u, \mathrm{div}\ \tilde{A} u) - 2(u, Bu) + 2(u, F) \\ &= (u, (\mathrm{div}\ \tilde{A} - (B + B^*))u) + 2(u, F) \\ c(u, u) &\leq E(t) \ \leq \ c^{-1}(u, u) \end{aligned}$$

Then (8.3) is a consequence of Gronwall's inequality.

§8.3 Local Smooth Solutions

Consider

$$\partial_t u + \nabla_x \cdot F(u) = S(u, x, t)$$

$$\begin{cases} \partial_t u + \sum_{j=1}^{m} \partial_{x_j} F_j(u) = S(u, x, t) \\ u(x, t = 0) = u_0(x) \end{cases} \tag{8.6}$$

$F(u) = (F_1(u), \cdots, F_m(u))$ smooth over D domain in $\mathbb{R}^n$.

Let D_1 be a bounded open subset of D, $D_1 \subset\subset D \Leftrightarrow \bar{D}_1 \subset D$,

$$u_0(x) \in \bar{D}_1 \tag{8.7}$$

Question: If $u_0 \in H^s(\mathbb{R}^m)$, $S(u_0, x, t) \in H^s$, $s > \frac{m}{2} + 1$. Then can we find $u(x,t) \in C^1([0,T] \times \mathbb{R}^m)$?

Definition 8.4 The system (8.6) is said to be admit a convex entropy extension if $\exists$ a convex entropy $\eta(u)$ with corresponding entropy flux $q(u) = (q_1(u), \cdots, q_m(u))$ such that for all smooth solutions $u(x,t)$ to the system (8.6).

$$\partial_t \eta(u) + \nabla_x \cdot q(u) = \nabla \eta(u) \cdot S(u,x,t)$$

i.e.

$$\nabla_u q_j(u) = \nabla_u \eta(u) \cdot \nabla_u F_j(u), \qquad j = 1, \cdots, m$$

Remark 8.3 If the system in (8.3) admits a convex entropy extension, then it is symmetrizable. In term of entropy variable, $U = \nabla\eta(u)$, the system (8.6) is symmetric.

For smooth solution, the system (8.6) is equivalent to

$$\partial_t u + \sum_{j=1}^{m} A_j(u)\partial_{x_j} u = S(u, x, t)$$

$$A_j(u) = \nabla_u F_j(u), \qquad j = 1, \cdots, m; \quad n \times n \text{ matrix}$$

So instead of considering (8.1), we will consider the following Cauchy problem

$$\begin{cases} A_0(u)\,\partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = S(u, x, t) \\ u(x, t = 0) = u_0(x) \end{cases} \tag{8.8}$$

where $\tilde{A} = (A_0, A_1, \cdots, A_m)$ satisfies the property that

$$A_0 > 0, \quad A_j^* = A_j, \quad j = 0, 1, \cdots, m \tag{8.9}$$

Notations:

$$H^s(\mathbb{R}^m) = \left\{ u \in L^2(\mathbb{R}^m), \text{ such that } ||u||_s^2 = \int_{\mathbb{R}^m} \sum_{|\alpha| \le s} |D^\alpha u(x)|^2 \, dx < \infty \right\}$$

$$C([0,T);\ H^s(\mathbb{R}^m)) = \left\{ u(x,t);\ u(\cdot,t) \in H^s,\ |||u|||_{s,T} = \max_{0 \le t \le T} ||u(\cdot,t)||_s < \infty \right\}$$

So the basic well-posedness theory is the

Theorem 8.2 Assume that

(1) (8.8) is symmetric, (8.9) holds.

(2) $u_0 \in H^s, \; s > \frac{m}{2} + 1, \; u_0(x) \in \bar{D}_1 \subset\subset D, \; \forall x.$

Then

(i) $\exists T = T(||u_0||_s, D_1)$ such that the Cauchy problem (8.8) has a unique classical solution $u(x,t) \in C^1([0,T] \times \mathbb{R}^m)$. With the properties that

$$u(x,t) \in \bar{D}_2 \subset D, \quad \forall (x,t) \in \mathbb{R}^m \times [0,T]$$

$$u(x,t) \in C([0,T], H^s) \cap C^1([0,T], H^{s-1}) \tag{8.10}$$

(ii) (Continuation principle) Let T^* be the maximal time of existence of regular solution as in (i). Suppose $T_* < +\infty$. Then, either

$$\overline{\lim}_{t\to T_*} \left(|Du(\cdot,t)|_{L^\infty} + |\partial_t u(\cdot,t)|_{L^\infty}\right) = +\infty \tag{8.11}$$

(shock formation)

or for any compact subset $k \subset\subset D$, then $u(\cdot,t)$ escapes from k as $t \to T_*^-$ (shell singularity).

Remark 8.4 There are two approaches. One is by T. Kato, ARMA (1952) p.181-205. Another one is due to P. Lax, elementary iteration scheme.

Proposition 8.1 Under the same assumptions in Theorem 8.2, there exists a unique classical solution $u(x,t) \in C^1(\mathbb{R}^m \times [0,T])$ to the problem (8.8) such that

$$u \in L^\infty([0,T];\ H^s(\mathbb{R}^m)) \cap C_w([0,T];\ H^s(\mathbb{R}^m)) \cap Lip([0,T];\ H^{s-1}) \quad (8.12)$$

Remark 8.5 $C_w([0,T];\ H^s(\mathbb{R}^m))$ *means continuous in time with values in* H^s *by weak topology, i.e.* $u \in C_w([0,T];\ H^s) \Leftrightarrow [u(s),\varphi]$ *is continuous on* $[0,T]$ *for any given* $\varphi \in H^{-s}$.

Proof of Proposition 8.1: The uniqueness is a simple consequence of the energy principle, so we omit it. We will concentrate on the existence and regularity.

Let $J_\varepsilon(x)$ be a Friedrichs mollifier, i.e. $J_\varepsilon(x) = \varepsilon^{-m} j(\frac{x}{\varepsilon})$, $j \in C_0^\infty(\mathbb{R}^m)$ supp $j \subset B_1(0)$, $\int_{\mathbb{R}^m} j(x)dx = 1$, $j \geq 0$.

$$\forall u \in H^s(\mathbb{R}^m),$$
$$J_\varepsilon u(x) = J_\varepsilon * u(x) = \int_{\mathbb{R}^m} J_\varepsilon(x-y)u(y)dy,$$
$$J_\varepsilon u \in H^s(\mathbb{R}^m) \cap C^\infty$$

Facts:
(1) $||J_\varepsilon u - u||_s \to 0$ as $\varepsilon \to 0+$.
(2) $||J_\varepsilon u - u||_0 \leq \hat{C}\varepsilon ||u||_1$, $\varepsilon \leq \varepsilon_0$, $\hat{C}$ is a generic positive constant.

Step 1: Preparation of Initial data

Setting

$$\varepsilon_k = 2^{-k}\varepsilon_0, \qquad u_0^k = J_{\varepsilon_k} u_0, \quad k = 0, 1, 2, \cdots \tag{8.13}$$

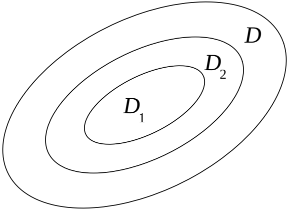
ε_0 is a suitably small positive constant defined later.

$$u_0 \in \bar{D}_1 \subset D$$

Thus one can choose another compact subset D_2 such that

$$\bar{D}_1 \subset\subset D_2, \qquad \bar{D}_2 \subset\subset D \tag{8.14}$$

Claim: One can choose R and ε_0 such that



(a) $$||u - u_0^0||_s \leq R \Rightarrow u \in \bar{D}_2 \quad (8.15)$$

(b) $$||u_0 - u_0^k||_s \leq C\frac{R}{4}, \quad k = 0, 1, 2, 3, \cdots \quad (8.16)$$

here $C(\leq 1)$ such that

$$CI \leq A_0(u) \leq C^{-1}I, \qquad \forall u \in \bar{D}_2 \tag{8.17}$$

By sobolev's imbedding's theorem, $|f|_{L^\infty} \leq C_s||f||_s$.

$$\begin{aligned} ||u - u_0||_s &\leq ||u - u_0^0||_s + ||u_0^0 - u_0||_s \\ &= ||u - u_0^0||_s + ||J_{\varepsilon_0} u_0 - u_0||_s \end{aligned}$$

Step 2: Iteration Scheme (By induction)

Set

- $u^0(x,t) = u_0^0(x)$.
- suppose $u^j(x,t)$ has been defined for $j = 0, 1, \cdots, k$, then we define $u^{k+1}(x,t)$ to be the solution to the following problem

$$\begin{cases} A_0(u^k)\partial_t u^{k+1} + \sum_{j=1}^{m} A_j(u^k)\,\partial_{x_j} u^{k+1} = S(u^k, x, t) \\ u^{k+1}(x, t=0) = u_0^{k+1}(x) \end{cases} \tag{8.18}$$

By the linear theory, (8.18) has smooth classical solution $u^{k+1}(x,t)$ defined on $\mathbb{R}^m \times [0, T_{k+1}]$ where T_{k+1} is such that

$$||u^{k+1} - u_0^0||_{s,T_{k+1}} \leq R \tag{8.19}$$

<u>Two main tasks</u>:

- one has to find a time interval $[0, T_*]$ such that all $u^k(x,t)$ can be defined $\mathbb{R}^m \times [0, T_*]$, i.e. $T_{k+1} \geq T_*$, $T_* > 0$, $k = 0, 1, \cdots$.
- $u^k(x,t) \to u(x,t)$ in appropriate topology.

Step 3: A priori estimate - boundedness in higher norm

Lemma 8.1 There exists $L > 0$, and $T_* > 0$, independent of k, such that for all $k = -1, 0, 1, 2, \cdots$

$$|||u^{k+1} - u_0^0|||_{s,T_*} \leq R \tag{8.20}$$

$$|||\partial_t u^{k+1}|||_{s-1,T_*} \leq L \tag{8.21}$$

Proof: Set $w^{k+1} = u^{k+1} - u_0^0$, then

$$\begin{cases} A_0(u^k)\,\partial_t w^{k+1} + \sum_{j=1}^{m} A_j(u^k)\partial_{x_j} w^{k+1} = S^k \\ w^{k+1}(x, t = 0) = u_0^{k+1}(x) - u_0^0(x) = w_0^{k+1}(x) \end{cases} \tag{8.22}$$

$$S^k = S(u^k, x, t) - \sum_{j=1}^{m} A_j(u^k)\,\partial_{x_j} u_0^0 \tag{8.23}$$

Remark 8.6 The key estimate is (8.20), since the temporal estimate (8.21) will follow from the system (8.18) with the help of Moser-type calculus inequality.

Obviously, $w^0 \equiv 0$, (8.20) holds trivially.

By inductive assumption, (8.20) holds true for u^k. For some T_* to be chosen, then $u^k \in \bar{\mathcal{D}}_2$.

So we can consider the following problem

$$\begin{cases} A_0(u)\,\partial_t w + \sum_{j=1}^m A_j(u)\,\partial_{x_j} w = S(u,x,t) \\ w(x,t=0) = w_0 \in \bar{D}_2 \end{cases} \tag{8.24}$$

$u \in C^\infty$, $w \in C^\infty$, $u \in \bar{D}_2$

Since $|||w|||_{s,T_*} = \max_{0\le t\le T_*} ||w(\cdot,t)||_s$, we need only to estimate $||D^\alpha w(\cdot,t)||^2 \ \forall\ 1 \le |\alpha| \le s,\ t \in [0, T_*]$.

Set $w_\alpha = D^\alpha W$, $|\alpha| \le s$. Then it follows from (8.24) that

$$\begin{cases} A_0\,\partial_t w_\alpha + \sum_{j=1}^{m} A_j\,\partial_{x_j} w_\alpha = A_0(u) D^\alpha(A_0^{-1} S) + S_\alpha = f_\alpha \\ S_\alpha = \sum_{j=1}^{m} A_0\left[(A_0^{-1}A_j)(u)\,\partial_{x_j} w_\alpha - D^\alpha((A_0^{-1}A_j)\partial_{x_j} w)\right] \\ w_\alpha(x, t=0) = D^\alpha w_0(x) \end{cases} \tag{8.25}$$

Claim: $\exists \bar{C} = \bar{C}(D_2, |||u|||_{s,T_*}, R, s)$ such that

$$\left(\sum_{1\leq|\alpha|\leq s} ||S_\alpha||_0^2\right) + \left(\sum_{|\alpha|\leq s} ||A_0 D^\alpha (A_0^{-1} S)||_0^2\right) \leq \bar{C}\left(1 + ||w||_s^2\right) \quad (8.26)$$

Then applying the energy inequality

$$E_\alpha(t) \leq \exp\left\{\frac{1}{2} C^{-1} |\text{div } A|_{L^\infty} T_*\right\} \left(E(0) + \int_0^{T_*} ||f_\alpha||_0^2 \, dt\right)$$

Sum them up, then

$$C||w(t)||_s^2 \leq \exp\left\{C^{-1} |\text{div } A|_{L^\infty} \; T_*\right\} \left(\bar{C}||w(0)||_s^2 + \int_0^{T_*} \left(1 + ||w(t)||_s^2\right) ds\right)$$

Now Grownwall inequality implies that

$$|||w|||_{s,T_*} \leq C^{-1} \exp\left\{\tilde{C}(1+L)T_*\right\}\left(||w_0||_s + \hat{C}T_*\right)$$
$$||w_0|| = ||u_0^{k+1} - u_0^0||_s \leq ||u_0^{k+1} - u_0||_s + ||u_0^0 - u_0||_s \leq C\frac{R}{4} + C\frac{R}{4} = \frac{CR}{2}$$
$$|||w|||_{s,T_*} \leq \exp\left(\tilde{C}(1+L)T_*\right)\left(\frac{R}{2} + \hat{C}T_*\right) \leq R$$

Note that T_*, L are independent of time.

It remains to prove the claim (8.26). To this end, we need some elementary Moser-type calculus inequalities.

Proposition 8.2 The follow facts hold

(1) If $u, v \in H^s$, $s > \frac{m}{2}$, then $uv \in H^s$.

$$||uv||_{H^s} \leq C_s \, ||u||_s \, ||v||_{H^s}$$

(2) If $u, v \in H^s \cap L^\infty$, then $u \cdot v \in H^s$.

$$||D^\alpha(uv)||_0 \leq C_s \left(|u|_{L^\infty} \, ||D^s u||_0 + |v|_{L^\infty} \, ||D^s u||_0\right)$$

for $1 \leq |\alpha| \leq s$

(3) $u \in H^s$, $Du \in L^\infty$, $v \in H^{s-1} \cap L^\infty$, and $|\alpha| \leq s$.

$$||D^\alpha(uv) - uD^\alpha v||_0 \leq C_s \left(|Du|_{L^\infty} \, ||D^{s-1} v||_0 + |v|_{L^\infty} \, ||D^s u||_0\right)$$

(4) Assume that $G(u)$ is a smooth function on a domain D, and furthermore, u is a continuous function of (x,t) such that $u(x,t) \in \bar{D}_1 \subset\subset D$ and $u \in H^s \cap L^\infty$. Then for $s \geq 1$,

$$||D^s G(u)||_0 \leq C_s \left| \frac{\partial G}{\partial u} \right|_{s-1,\bar{D}_1} ||D^s u||_0$$

$$\left| \frac{\partial G}{\partial u} \right|_{s-1,\bar{D}_1} \quad \text{is} \quad C^{s-1}(\bar{D}_1)\text{-norm}$$

Remark 8.7 Proposition 8.2 is called Moser-type calculus inequalities on Sobolev spaces, which are the consequences of the well-known Gagliardo-Nirenberge inequality:

For any $u \in H^s(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, $|D^i u|_{L^{2\frac{s}{i}}} \leq C_s |u|_{L^\infty}^{1-\frac{i}{s}} \, ||D^s u||_0^{\frac{i}{s}}$, $0 \leq i \leq s$.

Proof of the Claim: $\forall \alpha$, $1 \leq |\alpha| \leq s$.

$$\begin{aligned} ||A_0(u)\, D^{\alpha}(A_0^{-1}(u)S)||_0^2 &\leq C^{-1}\, ||D^{\alpha}(A_0^{-1}(u)\, S(u)||_0^2 \\ &\leq \hat{C}||u||_s^2 \leq \bar{C} \end{aligned}$$

$$\begin{aligned} & \sum_{1\leq|\alpha|\leq S} ||S_\alpha||_0^2 \\ \leq & \sum_{1\leq|\alpha|\leq S} ||A_0(u)\left[(A_0^{-1}\, A_j)(u) D^{\alpha}\, \partial_{x_j} w\right] - D^{\alpha}(A_0^{-1}\, A_j\, \partial_{x_j} w)||_0^2 \\ \leq & \ C^{-1}\left(||D(A_0^{-1}A_j)||_{L^\infty}\ ||D^{s-1}\, \partial_{x_j} w||_0 + |\partial_{x_j} w|_{L^\infty}\ ||D^s(A_0^{-1}A_j)||_0\right)^2 \\ \leq & \ C||w||_s^2 \end{aligned}$$

Step 4: Convergence of $u^k(x,t)$ (Contraction in lower norm estimate)

Idea: We need to find a norm $||\cdot||$ such that

$$\begin{array}{rlcl} & ||u^k - u|| \to 0 & \text{as} & k \to +\infty \\ \text{and} & A_j(u^k) \to A_j(u) & & j = 0, 1, 2, \cdots, m \\ & \nabla u^{k+1} \to \nabla u & \text{as} & k \to \infty \end{array}$$

Lemma 8.2 (Contraction in Lower-norm) $\exists T_{**} \in (0, T_*]$ and a sequence $\{\beta_k\}$ such that

$$|||u^{k+1} - u^k|||_{0,T_{**}} \le \alpha\, |||u^k - u^{k-1}|||_{0,T_{**}} + |\beta_k|$$

with $\alpha < 1$, $\sum_{k=0}^{\infty} |\beta_k| < +\infty$.

Proof of Lemma 8.2: Note that $u^{k+1} - u^k$ satisfies

$$\begin{cases} A_0(u^k)\,\partial_t(u^{k+1} - u^k) + \sum_{j=1}^{m} A_j(u^k)\,\partial_{x_j}(u^{k+1} - u^k) = g_k \\ (u^{k+1} - u^k)(x, t = 0) = u_0^{k+1} - u_0^k \end{cases}$$

$$g_k = S(u^k, x, t) - S(u^{k-1}, x, t) - \sum_{j=0}^{m}(A_j(u^k) - A_j(u^{k-1}))\,\partial_{x_j} u^k$$

Then the standard energy estimate

$$\begin{aligned} |||u^{k+1} - u^k|||_{0,T} &\leq C^{-1} \exp\{\tilde{C}\,T\}\{||u_0^{k+1} - u_0^k||_0 + T|||u^k - u^{k-1}|||_{0,T}\} \\ ||u_0^k - u_0||_0 &\leq C \cdot \varepsilon_k\,||u_0||_1 \quad \varepsilon_k = \varepsilon_0 2^{-k} \end{aligned}$$

It follows from Lemma 8.2 that

$$\exists u \in C([0, T_{**}], L^2(\mathbb{R}^m))$$

such that

$$|||u^k - u|||_{0,T_{**}} \to 0 \quad \text{as} \quad k \to \infty$$

Combining Lemma 8.1 with Lemma 8.2,

$$|||u^k|||_{s,T_{**}} + |||\partial_t u^k|||_{s-1,T_{**}} \leq \hat{C}$$

$$u^k \in \bar{D}_2$$

Furthermore, $u \in L^\infty([0, T_{**}]; H^s)$.

It follows from interpolation inequality

$$||w||_{s'} \leq C||w||_0^{1-\frac{s'}{s}} \; ||w||_s^{\frac{s'}{s}}$$

that

$$u^k \to u \quad \text{in} \quad C([0,T); H^{S'}) \quad \frac{m}{2} + 1 \leq s' < s$$

and

$$u \in C^0([0, T_{**}] \times \mathbb{R}^m)$$

and

$$u \in C([0, T_{**}]; C^1(\mathbb{R}^m))$$

$\partial_t u^k \to \partial_t u$ in $C([0, T_{**}]; C(\mathbb{R}^m))$ by using the equation, and immediately

$$u \in C^1([0, T_{**}] \times \mathbb{R}^m)$$

Therefore u is a classical solution of the Cauchy problem.

We need to show

$$u \in C_w([0, T_{**}]; H^s(\mathbb{R}^m)) \cap Lip([0, T_{**}], H^{s-1}(\mathbb{R}^m))$$

i.e. for $\forall \varphi \in (H^s(\mathbb{R}^m))' = H^{-s}(\mathbb{R}^m)$

$$\langle u(t), \varphi \rangle \quad \text{is continuous on} \quad [0, T_{**}]$$

Note the following facts,

(1) $H^{-s'}$ is dense in H^{-s}, $s' < s$.

(2) Since $u^k \to u$ in $C([0, T_{**}]; H^{s'}(\mathbb{R}^m))$, $\langle u^k, \tilde{\varphi}\rangle$ converges uniformly on $[0, T_{**}]$ for any $\varphi \in H^{-s'}$.

(3) $|||u^k|||_{s,T_{**}} \leq R + ||u_0^0||_s$.

Then (1), (2), (3) implies that $\langle u^k(t), \varphi\rangle$ converges uniformly to

$$\langle u(t), \varphi\rangle \quad \text{on} \quad [0, T_{**}]$$

Therefore $\langle u(t), \varphi\rangle$ is continuous on $[0, T_{**}]$.

$$\begin{aligned} & \langle u^k(t), \varphi\rangle - \langle u(t), \varphi\rangle \\ = \; & \langle u^k(t), \hat{\varphi}\rangle - \langle u(t), \hat{\varphi}\rangle + \langle u^k(t), \varphi - \hat{\varphi}\rangle + \langle u(t), \varphi - \hat{\varphi}\rangle \end{aligned}$$

This finishes the proof of Proposition 8.1.

Proposition 8.3 Let u be the classical solution in Proposition 8.1 satisfying

$$u(x,t) \in \bar{D}_2$$

and

$$u \in C_w([0, T_{**}]; H^s(\mathbb{R}^m)) \cap Lip([0, T_{**}]; H^{s-1}(\mathbb{R}^m))$$

Then

$$u \in C([0, T_{**}]; H^s(\mathbb{R}^m)) \cap C^1([0, T_{**}]; H^{s-1}(\mathbb{R}^m)) \qquad (8.27)$$

Proof: Weak implies strong by using the equations and the energy estimate.

It suffices to show that

$$||u_0||^2_{s,A_0(0)} \geq \overline{\lim}_{t\to 0+} \; ||u(t)||^2_{s,A_0(0)} = \overline{\lim}_{t\to 0+} \; ||u(t)||_{s,A_0(t)}$$

where

$$||u||^2_{s,A_0(t)} = \sum_{|\alpha|\leq s} \int_{\mathbb{R}^m} < D^\alpha u, A_0(u)\, D^\alpha u > dx$$

Recall that

$$u(x,t) \in \mathcal{D}_2 \subset\subset \mathcal{D}$$

$$CI \leq A_0(u(x,t)) \leq C^{-1} I, \quad 0 < C < 1$$

so

$$C||u(t)||_s^2 \leq ||u(t)||_{s,A_0(t)}^2 \leq C^{-1}||u(t)||_s^2$$

$|| \cdot ||_{s,A_0(t)}$ defines an equivalent norm on H^s.

Since A_0 is smooth enough, $A_0(u(x,t)) \in C^1$ where $A_0(0) = A_0(u_0(x))$.

$$u \in C_w([0, T_{**}], H^s(\mathbb{R}^m)),$$

so

$$u(\cdot, t) \rightharpoonup u_0(\cdot) \quad \text{as} \quad t \to 0+$$

therefore

$$u(\cdot, t) \to u_0(t) \quad \text{strongly in} \quad H^s(\mathbb{R}^m)$$

iff

$$||u_0||_{s,A_0(0)} \geq \overline{\lim}_{t\to 0+} \; ||u(t)||_{s,A_0(t)}$$

thus $u(\cdot, t)$ is continuous from right at $t = 0$.

This argument applies to each $t_0 \in [0, T_{**}]$, so $u(\cdot, t)$ is continuous from right at every $t_0 \in [0, T]$. On the other hand, the system (8.3) is hyperbolic. So it is time-reversible, the same argument implies $u(\cdot, t)$ is continuous from left at every $t_0 \in [0, T_{**}]$

$$A_0\,\partial_t u + \sum_{j=1}^{m} A_j\,\partial_{x_j} u = S(u, x, t)$$

Hence, $u(\cdot, t)$ is continuous at $[0, T]$.

To show (8.27), we have a lemma,

Lemma 8.3 Let u be the classical solution constructed in $[0, T_{**}]$. Then there exists a function $f(t) \in L^1([0, T_{**}])$ such that

$$||u(t)||^2_{s,A_0(t)} \leq ||u_0||^2_{s,A_0(0)} + \int_0^t f(s)ds \qquad (8.28)$$

Let us assume Lemma 8.3 holds, then taking limits $t \to 0+$ in (8.28) immediately, we obtain

$$\overline{\lim}_{t\to 0+} ||u(t)||^2_{s,A_0(t)} \leq ||u_0||^2_{s,A_0(0)}$$

This is nothing but (8.27).

It remains to prove Lemma 8.3. Due to the uniqueness of classical solution, we can assume that $u(x,t)$ is the limit of the approximate solution $u^k(x,t)$.

$$u^k(x,t) \in C^\infty \cap H^s$$

with the uniform H^s-estimate in Lemma 8.1.

Set $u_\alpha^{k+1} = D^\alpha u^{k+1}$. Then as before,

$$A_0(u^k)\partial_t u_\alpha^{k+1} + \sum_{j=1}^m A_j(u^k)\,\partial_{x_j} u_\alpha^{k+1} = S_\alpha$$

where

$$S_\alpha = A_0(u^k) D^\alpha (A^{-1}(u^k) S(u^k, x, t)) + F_\alpha$$

$$F_\alpha = \begin{cases} 0 \\ \displaystyle\sum_{j=1}^{m} A_0(u^k) \left[A_0^{-1}(u^k) A_j(u^k) \partial_{x_j} \, u_\sigma^{k+1} - D^\alpha (A_0^{-1}(u^k) A_j(u^k) \partial_{x_j} \, u^{k+1}) \right] & (k \geq 1) \end{cases}$$

Thus the energy estimates yield

$$\begin{aligned} & \frac{d}{dt} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^m} (D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1}) \\ = \quad & \int_{\mathbb{R}^m} \sum_{|\alpha| \leq s} (\operatorname{div} \vec{A}(u^k) D^\alpha u^{k+1}, D^\alpha u^{k+1}) + 2 \int_{\mathbb{R}^m} \sum_{|\alpha| \leq s} (S_\alpha, D^\alpha u^{k+1}) dx \end{aligned} \tag{8.29}$$

Claim: The right hand side is in $L^\infty([0, T_{**}])$

$$\vec{A} = (A_0, A_1, \cdots, A_m) \qquad \text{(based on Lemma 8.1)}$$

Then

$$\frac{d}{dt} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^m} \langle D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1} \rangle \leq f(t)$$

hence

$$\begin{aligned} & \sum_{|\alpha| \leq s} \int_{\mathbb{R}^m} \langle D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1} \rangle dt \\ \leq \quad & \sum_{|\alpha| \leq s} \int_{\mathbb{R}^m} \langle D^\alpha u_0^{k+1}, A_0(u^k) D^\alpha u_0^{k+1} \rangle dx + \int_0^t f(s) ds \end{aligned}$$

Taking limit $k \to \infty$,

$$\begin{aligned}
& \overline{\lim}_{k\to\infty} \left(\sum_{|\alpha|\le s} \int_{\mathbb{R}^m} (D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1}) dx \right) \\
\le \quad & \overline{\lim}_{k\to\infty} \sum_{|\alpha|\le s} \int_{\mathbb{R}^m} (D^\alpha u_0^{k+1}, A_0(u_0^k) D^\alpha u_0^{k+1}) dx + \int_0^t f(s) ds \\
= \quad & ||u_0||_{s, A_0(0)} + \textstyle\int_0^t f(s) ds
\end{aligned}$$

By weak convergence of $u^k \rightharpoonup u$ in H^s, and $u^k \to u$ in $H^{s'}$, $s' > \frac{m}{2} + 1$, we have

$$\begin{aligned}
& \overline{\lim}_{k\to\infty} \left(\sum_{|\alpha|\le s} \int_{\mathbb{R}^m} (D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1}) dx \right) \\
\ge \quad & \sum_{|\alpha|\le s} \int_{\mathbb{R}^m} (D^\alpha u(t), A_0(u(t)) D^\alpha u(t)) dx
\end{aligned}$$

Continuation Principle

$$\begin{cases} A_0(u)\,\partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = S(u,x,t) \\ u(x,t=0) = u_0 \in H^s(\mathbb{R}^m) \end{cases}$$

where $s > \frac{m}{2} + 1$, $u \in \mathcal{D}_1 \subset\subset D_2$

$$\exists T = T(S, ||u_0||_s) > 0$$

$$u \in C([0,T]; H^s(\mathbb{R}^m)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^m))$$

how large is T?

Let $[0, T]$ be the maximum interval of existence of such H^s solution. Then clearly

$$\begin{array}{lll} \text{either} & T = +\infty, & u \in ([0,\infty); H^s(\mathbb{R}^m)) \\ \text{or} & T < +\infty, & \text{then} \end{array}$$

$$\lim_{t \to T^-} ||u(t)||_s = +\infty$$

Since, if otherwise, $\overline{\lim}_{t \to T^-} ||u(t)||_s < +\infty$.

Then

$$\begin{cases} A_0 \, \partial_t u + \sum_{j=1}^{m} A_j \, \partial_{x_j} u = S(u, x, t) \\ u(x, t = T - \varepsilon) = u|_{t = T - \varepsilon} \in H^s \end{cases}$$

Sharp Continuation Principle

Proposition 8.4 Assume that

(1) $u_0 \in H^s$, $s > \frac{m}{2} + 1$, $u_0 \in \mathcal{D}_1 \subset\subset \mathcal{D}$.

(2) Let T be given time $T > 0$.

Assume that $\exists$ constants C_1 and C_2 and a fixed open set $\mathcal{D}_2$ such that $\mathcal{D}_1 \subset\subset \mathcal{D}_2 \subset\subset \mathcal{D}$, so that on any interval of existence of H^s-solution in Theorem 8.2, $[0, T_{**}]$, $T_* \leq T$, the following a priori estimate hold.

(i) $|\text{div}\, \vec{A}|_{L^\infty} \leq C_1$ on $[0, T_*]$.

(ii) $|Du|_{L^\infty} \leq C_2$ on $[0, T_*]$.

(iii) $u(x, t) \in \bar{\mathcal{D}}_2 \quad \forall (x, t) \in \mathbb{R}^m \times [0, T_*]$.

Then

(a) u exists on $[0, T]$ such that
$u \in C([0, T]; H^s(\mathbb{R}^m)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^m))$.

(b) $|||u(t)|||_{s,T_*} \leq \exp\{(C_1 + C_2)CT\}\{||u_0||_s + C\}$, $\forall\, T_* \in [0, T]$,
C is a uniform constant.

Remark 8.8 If $[0, T]$ is a maximal interval of existence of H^s solution, and $T < +\infty$, then either $\lim_{t \to T_-} (|\partial_t u|_{L^\infty} + |\nabla u|_{L^\infty}) = +\infty$ or $u(x, t)$ escapes every compact subset of $\mathcal{D}$ as $\in \to T_-$.

Remark 8.9 Assume that

(1) $u_0 \in H^s$, $s > \frac{m}{2} + 1$.

(2) $u(x,t)$ is a classical solution to (10.11), i.e. $u \in C^1(\mathbb{R}^m \times [0,T])$.

Then, on the same interval $[0,T]$, $u \in C([0,T]; H^s(\mathbb{R}^m)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^m))$. In particular, if

(i) $u_0 \in \cap_s H^s$;

(ii) $u \in C([0,T]; H^s(\mathbb{R}^m))$ for some $s_0 > \frac{m}{2} + 1$ and u is a solution to (8.8).

Then $u \in C^\infty(\mathbb{R}^m \times [0,T])$.

Proof of Proposition 8.4: By the standard continuity argument, it suffices to prove the a priori estimate in (b). Let $u(x,t)$ be classical H^s-solution to (8.8) and satisfies (i)-(iii).

$$\begin{cases} A_0(u)\,\partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = S(u,x,t) \\ u(x,t=0) = u_0(x) \in \mathcal{D}_1 \subset\subset \mathcal{D} \end{cases}$$

(iii) implies that

$$CI \leq A_0(u(x,t)) \leq C^{-1} I$$

Set $u^\alpha = D^\alpha u$,

$$A_0\, \partial_t u^\alpha + \sum_{j=1}^{m} A_j\, \partial_{x_j} u^\alpha = S_\alpha$$

$$S_\alpha = A_0 D^\alpha (A_0^{-1} S) + F_\alpha$$

$$F_\alpha = -\sum_{j=1}^{m} A_0(u)[D^\alpha (A_0^{-1} A_j\, \partial_{x_j} u) - A_0^{-1} A_j\, \partial_{x_j} u^\alpha]$$

$$F_\alpha = 0 \quad \text{for} \quad \alpha = 0$$

For $1 \leq |\alpha| \leq s$,

$$\begin{aligned}
& \sum_{1\leq|\alpha|\leq s} ||F_\alpha||_0 \\
\leq \quad & \sum_{\substack{1\leq|\alpha|\leq s \\ 1\leq j\leq m}} C^{-1} \left(|D(A_0^{-1}A_j)|_{L^\infty} \, |D^{s-1}\,\partial_{x_j} u|_0 + |\partial_{x_j} u|_{L^\infty} \, |D^s(A_0^{-1}A_j)|_0 \right) \\
\leq \quad & C \cdot C_2 ||D^s u||_0
\end{aligned}$$

$$\sum_{|\alpha|\leq s} ||A_0 D^\alpha (A_0^{-1} S)||_0 \leq C||u||_s$$

Then the uniform estimate in (b) follows from this and energy principle.

Remark 8.10 This completes the local well-posedness of classical solution to the Cauchy problem

$$\begin{cases} A_0(u)\,\partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = S \\ u(x, t=0) = u_0 \in H^s(\mathbb{R}^m) \quad s > \frac{m}{2} + 1 \end{cases}$$

$\lim_{|x|\to\infty} u(x,t) = \bar{u}$

Local energy principle and finite speed of Propagation

Consider

$$\begin{cases} A_0(x,t)\,\partial_t u + \sum_{j=1}^{m} A_j(x,t)\,\partial_{x_j} u + B(x,t)u = F(x,t) \\ u(x,t=0) = u_0(x) \end{cases} \tag{8.30}$$

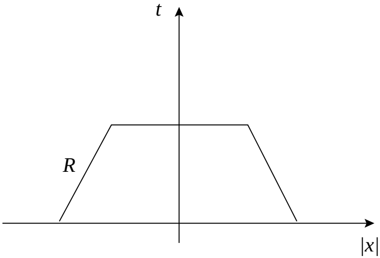
where

$$A_j^*(x,t) = A_j(x,t), \qquad j = 0, 1, \cdots, m. \tag{8.31}$$

$$CI \le A_0(x,t) \le C^{-1} I, \qquad (C \le 1) \tag{8.32}$$

$$\max_{\substack{|w|=1\\(x,t)}} \left| \sum_{j=1}^{m} \langle A_j(x,t) w_j V, V \rangle \right| \leq \frac{D}{2C} |V|^2$$

$$R = \frac{D}{2C}$$



Proposition 8.5 (Local energy principle) Let u be a classical solution to (8.30). Then it follows that

$$\begin{aligned} & \int_{|x-y|\le d} (A_0 u, u)(T) dx \\ \le \ & \int_{|x-y|\le d+RT} (A_0 u_0, u_0) dx \\ & + \int_0^T \int_{|x-y|\le d+R(T-t)} |2(F,u) + (\operatorname{div} \vec{A} u, u) + ((B+B^*)u, u)| dx dt \end{aligned}$$

Proof: By direct computation, using the symmetry of $\vec{A}$

$$\frac{\partial}{\partial t}(u^* A_0 u) + \sum_{j=1}^{m} \partial_{x_j}(u^* A_j u) = u^* \operatorname{div} \vec{A} u + u^* B u + u^* B^* u + 2u^* F$$

Then integrate on the trapezoid, using the Gauss formula,

Definition 8.5 (Uniformly Local Sobolev Space)
Let $u \in H^s_{loc}(\mathbb{R}^m)$, then u is said to be in the uniformly local Sobolev space $H^s_{ul}(\mathbb{R}^m)$. If

$$\max_{y \in \mathbb{R}^m} ||\theta_{d,y} u||_s = \tilde{|}|u|\tilde{|}_{s,d} < +\infty \quad \text{for some } d$$

where

$$\begin{aligned} \theta_{d,y} &= \theta\left(\frac{|x-y|}{d}\right) \\ \theta(r) &= \begin{cases} 1 & \text{if} \quad r < \dfrac{1}{2} \\ 0 & \text{if} \quad r > 1 \end{cases} \qquad 0 \leq \theta \leq 1 \qquad \theta \in C^\infty(\mathbb{R}^+) \end{aligned}$$

Remark 8.11 $\tilde{||}\cdot\tilde{||}_{s,d}$ are equivalent norms for H^s_{ul} for different d and

$$\tilde{||}u\tilde{||}_{s,d_1} \leq C\tilde{||}u||_{s,d_2}$$
$$0 < d_- \leq d_1, \qquad d_2 \leq d_+ < +\infty$$

Remark 8.12 In the uniform local Sobolev space H^s_{ul}, the local energy principle

$$\begin{aligned} & \tilde{||}u\tilde{||}_{0,d}(T) \\ \leq \quad & C^{-1}\exp\left(\frac{1}{2}C^{-1}|\mathrm{div}\,\vec{A}+(B+B^*)|_{L^\infty}T\right)\left(\tilde{||}u_0\tilde{||}_{0,2d+RT}+\int_0^T\tilde{||}F\tilde{||}_{0,2d+R(T-t)}dt\right) \end{aligned}$$

Remark 8.13 If $\lim_{|x|\to+\infty} u(x,t) = \bar{u}$, then $u \in H^s_{ul}$.

Other interesting uniform local spaces are used to handle the cases such that

$$u(x,t) = u(x_1,t)\text{: planary functions,}$$

$$u\text{ : periodic function.}$$

Theorem 8.3 Assume that

(1) $u_0 \in H^s_{ul}(\mathbb{R}^m)$, $s > \frac{m}{2} + 1$

(2) $u_0 \in \bar{\mathcal{D}}_1 \subset\subset D$

Then there exists $T = T(\tilde{\|} u_0 \tilde{\|}_{s,d}, \mathcal{D}_1)$ such that the Cauchy Problem (8.8) has a unique solution $u \in C^1([0, T] \times \mathbb{R}^m)$ with the properties

(i) $u(\cdot, t) \in \bar{\mathcal{D}}_2$, $\bar{\mathcal{D}}_1 \subset\subset \mathcal{D}_2 \subset\subset \mathcal{D}$

(ii) $u \in C([0, T]; H^s_{\text{loc}}(\mathbb{R}^m)) \cap C^1([0, T]; H^{s-1}_{\text{loc}}(\mathbb{R}^m))$

(iii) $u \in L^\infty([0, T]; H^s_{ul})$

Theorem 8.4 (Continuation Principle) Assume that

(1) $u_0 \in H^s_{ul}(\mathbb{R}^m)$, $s > \frac{m}{2} + 1$

(2) $T > 0$ be given constant

(3) $\exists$ fixed constants M_1 and M_2 and a fixed open set $\mathcal{D}_1$ with $\bar{\mathcal{D}}_1 \subset \mathcal{D}$ independent of $T_* \in [0, T]$ so that for any time interval $[0, T_*]$ of the local $H^s_{ul}(\mathbb{R}^m)$ solution, $T_* \leq T$, the following a priori estimates hold

(i) $|\mathrm{div}\, \vec{A}|_{L^\infty} \leq M_1$, $0 \leq t \leq T_*$

(ii) $|Du|_{L^\infty} \leq M_2$, $0 \leq t \leq T_*$

(iii) $u(x, t) \in \bar{\mathcal{D}}_1$, $\forall (x, t) \in \mathbb{R}^m \times [0, T_*]$

Then the local regular solution exists on $[0, T]$ with $u \in C([0, T]; H^s_{\text{loc}}) \cap C^1([0, T]; H^{s-1}_{\text{loc}}) \cap L^\infty([0, T]; H^s_{ul})$. Furthermore, the local uniform energy estimate holds.

Remark 8.14 For one-dimensional theory

$$\begin{cases} \partial_t u + A(u)\,\partial_x u = S(u, x, t) & x \in \mathbb{R}^1, \quad u \in \mathbb{R}^n \\ u(x, t = 0) = u_0(x) \end{cases} \tag{8.8)'$$

Theorem 8.5 Assume that

(1) $u_0 \in C^1(\mathbb{R}^1)$ such that

$$||u_0||_{C^1} = |u_0|_{L^\infty} + |u_0'|_{L^\infty} < +\infty$$

(2) $u_0 \in \bar{\mathcal{D}}_1 \subset\subset D$

Then there exists $T = T(\bar{\mathcal{D}}_1, ||u_0||_{C^1}) > 0$ such that there exists a unique solution to $(8.8)'$ on $\mathbb{R}^1 \times [0, T]$. Furthermore, let T_* be the maximal length of the time interval $[0, T_*]$ of the existence of classical solution and $T_* < +\infty$. Then

either	$\lim_{t \to T_*} |\partial_x u(\cdot, t)|_{L^\infty} = +\infty$
or	$u(x, t)$ runs out of any compact subset of D as $t \to T_*-$

(Proof by characteristic method)

§8.4 Blow-up of Smooth Solutions and Formation of Shock Waves

$$\partial_t u + \sum_{j=1}^{m} \partial_{x_j} F_j(u) = 0$$

$$u(x, t = 0) = u_0(x)$$

$$u_0 \in H^s_{ul}(\mathbb{R}^m), \quad s > \frac{m}{2} + 1$$

First, we have a local solution, $u \in C^1(\mathbb{R}^m \times [0, T])$. Then

either $\quad T = +\infty, \quad$ i.e., $\quad \exists$ global in time regular solution

or $\quad$ maximal $T < +\infty \begin{cases} \text{either} & \lim_{t \to T^*} |\nabla_x u(\cdot t)|_{L^\infty} = +\infty \\ \text{or} & u \text{ runs out of every compact subset of } \mathcal{D} \end{cases}$

In particular, if $\mathcal{D} = \mathbb{R}^n$, then the second case implies

$$\lim_{t \to T_-} |u(\cdot, t)|_{L^\infty} = +\infty$$

Case 1: Formation of shock waves

Case 2: Shell singularity

Main Tasks in the Theory of Hyperbolic Conservation Laws

(1) Generally, shock waves form in finite time
(2) After formation of shock wave, how to extend the "solution" globally in time in a "unique" way
   - formation of shocks for scalar equation
   - formation of shocks for planar waves (One-dimensional Theory)
   - formation of singularity for 3-D compressible Euler equation

§8.4.1 Scalar equations

$$\begin{cases} \partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = 0 \qquad u \in \mathbb{R}^1 \\ u(x, t=0) = u_0(x) \end{cases} \tag{8.33}$$

where

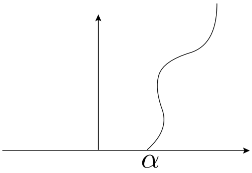
$$A_j(u) = \frac{dF_j(u)}{du}$$

If $m = 1$,

$$\begin{aligned} &\partial_t u + \partial_x f(u) = 0 \\ \text{or} \quad &\partial_t u + a(u)\,\partial_x u = 0 \end{aligned}$$

Its characteristic $x = x(t, \alpha)$ is defined to

$$\begin{cases} \dfrac{dx}{dt} = a(u(x(t,\alpha),t)) \\ x(t=0,\alpha) = \alpha \end{cases}$$



for any C^1-solution $u(x,t)$. Then

$$\frac{d}{dt} u(x(t,\alpha),t) = 0$$

$$u(x(t,\alpha),t) = u_0(\alpha)$$

Method 1: (Explicit formula)

In this case, $a(u(x(t,\alpha),t)) = a(u_0(\alpha))$

$$\begin{aligned} x &= \alpha + a(u_0(\alpha))t \\ u(x,t) &= u_0(x - a(u_0(\alpha))t) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad ||u(t,\cdot)||_{L^\infty} &= ||u_0||_{L^\infty} \\ \partial_x u(x,t) &= u_0'(\alpha)\frac{\partial \alpha}{\partial x} \\ &= u_0'(\alpha)\frac{1}{1 + \frac{d}{d\alpha}a(u_0(\alpha))t} \end{aligned}$$

If $\dfrac{d}{d\alpha}a(u_0(\alpha)) \geq 0$, then $|\partial_x u(x,t)| \leq ||u_0'(\alpha)||_{L^\infty}$.

Using the equation, $|\partial_t u|_{L^\infty} \leq C$.

$|Du|_{L^\infty} \leq M_1 < +\infty$, so there exists global smooth solution. If the above condition fails, then $\exists\, \alpha_0$ such that

$$\frac{d}{d\alpha} a(u_0(\alpha))|_{\alpha=\alpha_0} < 0$$

Then $u_0'(\alpha_0) \neq 0$, when

$$T \to T_* = -\frac{1}{\frac{d}{d\alpha} a(u_0(\alpha))} < +\infty$$

$$|\partial_x u(x,t)| = \left| \frac{u_0'(x)}{1 + \frac{d}{d\alpha} a(u_0(\alpha)) t} \right| \to +\infty \qquad \text{as} \quad t \to T_* -$$

In most cases, blow-up is proved by comparing some differential inequality about a functional involving u and ∇u with a Ricatti type equation

$$\frac{dy}{dt} = y^2$$

Method 2:

$$a'(u)(\partial_t u + a(u)\partial_x u) = 0$$

Then

$$\partial_t a(u) + a(u)\,\partial_x a(u) = 0$$

i.e.

$$\partial_t a(u) + \partial_x \left(\frac{a(u)^2}{2}\right) = 0$$

$w = a(u)$,

$$\partial_t w + \partial_x \left(\frac{1}{2} w^2\right) = 0$$

Differentiate the above equation with respect to x,

$$\partial_x \left(\partial_t w + \partial_x \left(\frac{w^2}{2}\right)\right) = 0$$

$$\Rightarrow \quad \partial_t(\partial_x w) + w\,\partial_x(\partial_x w) + (\partial_x w)^2 = 0$$

along the characteristic $x = x(t, \alpha)$

$$\frac{d}{dt} q(x(t), t) + q^2 = 0$$

where $q(x, t) = \partial_x w(x, t)$.

Solving this Ricatti equation

$$q(x, t) = \frac{q_0}{1 + t q_0}$$

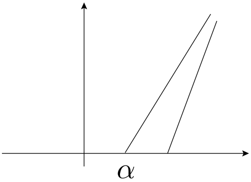
$$|q(\cdot, t)|_{L^\infty} < \infty \qquad \text{iff} \qquad q_0 \geq 0$$

$$q_0 = \frac{d}{d\alpha} a(u_0(\alpha))$$

Method 3: (Geometric)

$$x = x(t, \alpha)$$

$$\begin{cases} \dfrac{dx}{dt} = a(u(x(t,\alpha),t)) = a(u_0(\alpha)) \\ x(t=0,\alpha) = \alpha \end{cases}$$



If $a(u_0(\alpha))$ increases with respect to α, then wave expands, so there are no singularities.

$$\exists\, \alpha_0 > 0, \qquad \frac{d}{d\alpha} a(u_0(\alpha)) < 0$$

$\exists\, \alpha_1$ and α_2, $\alpha_1 < \alpha_2$ such that $a(u_0(\alpha_1)) > a(u_0(\alpha_2))$
$\Rightarrow$ wave compression.

For multidimensional case,

$$\begin{cases} \partial_t u + \sum_{j=1}^{m} A_j(u)\, \partial_{x_j} u = 0, \quad A_j(u) = \dfrac{d\, F_j(u)}{du} \\ A(u) = (A_l(u), \cdots, A_m(u)) \\ u(x, t = 0) = u_0(x) \end{cases} \tag{8.34}$$

We define the characteristic curve through initial point $\alpha = (\alpha_1, \cdots, \alpha_m)$ as $x = x(t, \alpha)$ satisfies

$$\begin{cases} \dfrac{\partial x}{\partial t} = A(u(x(t,\alpha),t)) \\ x(t=0,\alpha) = \alpha \end{cases}$$

where u is a C^1-regular solution to (8.34).

$$\begin{aligned} & \frac{d}{dt} u(x(t,\alpha),t) = \partial_t u + \sum_{j=1}^{m} A_j(u)\, \partial_{x_j} u = 0 \\ \Rightarrow \quad & u(x(t,\alpha),t) = u_0(\alpha) \\ \Rightarrow \quad & x = \alpha + A(u_0(\alpha))t \end{aligned}$$

Method 1: (Explicit formula)

$$u(x,t) = u_0(\alpha) = u_0(x - A(u_0(\alpha))t)$$

(1)

$$||u(\cdot,t)||_{L^\infty} = ||u_0||_{L^\infty} < +\infty$$

(2)

$$\nabla_x u(x,t) = \nabla_\alpha u_0(\alpha)\frac{\partial \alpha}{\partial x}.$$

It can be shown that (e.x.)

$$\nabla_x u(x,t) = \frac{\nabla_\alpha u_0(\alpha)}{1 + t\,\mathrm{div}_\alpha A(u_0(\alpha))}.$$

so

$$|\nabla_x u(x,t)| = \frac{|\nabla_\alpha u_0(\alpha)|}{1 + t \,\mathrm{div}_\alpha A(u_0(\alpha))|}$$

If $\mathrm{div}_\alpha A(u_0(\alpha)) \geq 0$, then there will be global smooth solution.

If $\exists\, \alpha_0$, such that $\mathrm{div}_\alpha A(u_0(\alpha))|_{\alpha_0} < 0$.

Set

$$T_* = -\frac{1}{\mathrm{div}_\alpha A(u_0(\alpha))} < +\infty$$

as $t \to T_*$, $||\nabla_x u(x,t)||_{L^\infty} \to \infty$ as $t \to T_*$.

Method 2: (Reduced to Ricatti equation)

$$\partial_{x_i}\left\{\left(\partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = 0\right)\right\}$$

$$\Rightarrow \quad \partial_t(\partial_{x_i} u) + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j}(\partial_{x_i} u) + \sum_{j=1}^{m}\sum_{i=1}^{m} A_j'(u)\,\partial_{x_j} u\,\partial_{x_i} u = 0$$

Multiply the both sides by $A_i'(u)$, and sum up from 1 to m,

$$\sum_{i=1}^{m} A_i'(u)\,\partial_t(\partial_{x_i} u) + \sum_{i,j=1}^{m} A_i'(u)\,A_j(u)\,\partial_{x_j}(\partial_{x_i} u)$$

$$+ \sum_{\substack{j=1\\ i=1}}^{m} A_j'(u)\,A_i'(u)\,\partial_{x_i} u\,\partial_{x_j} u = 0$$

Define $q(x,t) = \sum_{i=1}^{m} A_i'(u)\,\partial_{x_i} u = \mathrm{div}_x A(u)$.

$$\partial_t \left(\sum_{i=1}^{m} A_i'(u)\,\partial_{x_i} u \right) + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} \left(\sum_{i=1}^{m} A_i'(u)\,\partial_{x_i} u \right)$$

$$\left(\sum_{i=1}^{m} A_i''(u)\,\partial_t u\,\partial_{x_i} u + \sum_{j=1}^{m} A_j(u) \sum_{i=1}^{m} A_i''(u)\,\partial_{x_j} u\,\partial_{x_i} u \right) + q^2 = 0$$

$$\Rightarrow \quad \partial_t q + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} q + q^2 = 0$$

$$\Rightarrow \quad \frac{dq}{dt} + q^2 = 0$$

Therefore, $q(x,t) = \dfrac{q_0}{1 + q_0 t}$.

$$\text{div}_x\, A(u(x,t)) = \frac{\text{div}_\alpha\, A(u_0(\alpha))}{1 + \ \text{div}_\alpha\, A(u_0(\alpha))t}$$

If $\exists\, \alpha_0$ such that

$$\text{div}_\alpha\, A(u_0(\alpha)) < 0,$$

shock must form at

$$T_* = -\frac{1}{\text{div}_\alpha\, A(u_o(\alpha))}$$

Theorem 8.6 Assume that $u_0 \in H^s_{ul}(\mathbb{R}^m)$, $s > \frac{m}{2} + 1$, then the Cauchy problem (8.34) has a unique global regular solution iff

$$\operatorname{div}_\alpha A(u_0(\alpha)) \geq 0$$

Furthermore, if

$$\min \operatorname{div}_\alpha A(u_0(\alpha)) = m_0 < 0$$

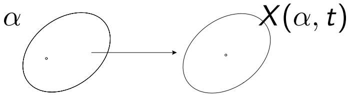
then shock wave must form at $T_* = -\frac{1}{m_0}$.

Remark 8.15 $H^s_{ul}(\mathbb{R}^m)$ can be replaced by C^1_b.

Remark 8.16 (Geometric meaning of the singularity)

Let $u(x,t)$ be regular on $\mathbb{R}^m \times [0, T]$,

Lagrangian map: $L : \alpha \mapsto X(\alpha, t)$



$$J(t,\alpha) = \det\left(\frac{\partial X}{\partial \alpha}\right)$$

$J(t,\alpha)$ measures the ration of the volume in the image to the volume initially along the characteristic curve $X(t,\alpha)$

$$\text{locally compression:} \quad \frac{d}{dt}J(t,\alpha) < 0$$

$$\text{locally expansion:} \quad \frac{d}{dt}J(t,\alpha) > 0$$

wave breaks down means infinite compression, i.e.

$$J(t,\alpha) \to 0 \qquad as \qquad t \to T_*$$

$$\begin{cases} \dfrac{\partial X(t,\alpha)}{\partial t} = A(u(x(t,\alpha),t)) \\ X(t=0,\alpha) = \alpha \end{cases}$$

Then $\frac{d}{dt}J(t,\alpha) = (\text{div}_x\, A(u(x(t,\alpha),t)))\, J(t,\alpha)$.

If $q(t,\alpha) = \mathrm{div}_x A(u(x(t,\alpha),t)) > 0$,

wave expands $\quad\Rightarrow\quad$ global existence of solution.

If $q(t,\alpha) < 0$, wave compressive.

In particular, if $q_0(\alpha_0) < 0$, shock must form.

Since

$$
\begin{aligned}
J(\alpha,t) &= \exp\int_0^t q(s,\alpha)ds \\
&= \exp\int_0^t \frac{q_0(\alpha)}{1+q_0(\alpha)s}ds \\
&= 1 + q_0(\alpha)t \\
&\to 0 \quad as \quad t \to T_* = -\frac{1}{q_0(\alpha)}
\end{aligned}
$$

§8.4.2 Plane waves and formation of shock waves

Given any direction $w \in \mathbb{R}^m$, $|w| = 1$, look for special

$$u(x,t) = U(x \cdot w, t)$$
$$\xi = x \cdot w, \qquad u(x,t) = U(\xi, t)$$
$$\partial_t u + \sum A_j \, \partial_{x_j} u = 0$$

$\Rightarrow$

$$\partial_t U + A(u,w) \, \partial_\xi U = 0$$

where

$$A(u,w) = \sum_{j=1}^{m} A_j(u) w_j$$

$$\begin{cases} \partial_t u + A(u,w) \, \partial_\xi u = 0, \quad t > 0, \quad \xi \in \mathbb{R}^1 \\ u(x, t=0) = u_0(\xi) \end{cases}$$

P. Lax, F. John, L. Hörmander.

§8.4.3 Shock wave formation in plane wave solution

$$\begin{cases} \partial_t u + \sum_{j=1}^{m} \partial_{x_j} F_j(u) = 0 \\ u(x, t=0) = u_0(x) \end{cases} \tag{8.35}$$

$u(x,t) = U(x \cdot w, t)$ for a given direction $w \in \mathbb{R}^m$, $|w| = 1$, $A_i(u) = \frac{\partial F_i}{\partial u}$.

Set $\xi = x \cdot w$,

$$\begin{cases} \partial_t u + A(u,w)\, \partial_\xi u = 0 \\ u(\xi, t=0) = u_0(\xi) = u_0(x \cdot w) \end{cases} \tag{8.36}$$

where $A(u,w) = \sum_{i=1}^{m} w_i A_i(u)$,

$$
\begin{array}{ccccccc}
\lambda_1(u,w) & \leq & \lambda_2(u,w) & \leq & \cdots & \leq & \lambda_n(u,w) \\
r_1(u,w), & & r_2(u,w), & & \cdots & & r_n(u,w) \\
l_1(u,w), & & l_2(u,w), & & \cdots & & l_n(u,w)
\end{array}
$$

$LR = I.$

Blow-up of simple waves

Let k be fixed, $1 \leq k \leq n$. Assume that $\bar{u}_0 \in D \subset \mathbb{R}^n$. Regard $r_k(u)$ as a vector field on D. As we can look the integral curve of $r_k(u)$ through $\bar{u}_0$, i.e.

$$
\begin{cases}
\dfrac{dU_k(\sigma)}{d\sigma} = r_k(U_k(\sigma)) \\
U_k(\sigma = 0) = \bar{u}_0
\end{cases}
\tag{8.37}
$$

$\exists\, \sigma_\pm$, $\sigma_- < \sigma < \sigma_+$ such that (8.37) has a smooth solution $U_k(\sigma)$, $\sigma \in (\sigma_-, \sigma_+)$.

$U_k(\sigma)$ is called a k-th wave curve through $\bar{u}_0$.

Next, solve the following initial value problem

$$\begin{cases} \partial_t \sigma + \lambda_k(U_k(\sigma))\partial_\xi \sigma = 0 & \xi \in \mathbb{R}^1, \quad t > 0 \\ \sigma(t=0) = \sigma_0(\xi) & \sigma_- < \sigma_0(\xi) < \sigma_+, \quad \forall \xi \in \mathbb{R}^1 \end{cases} \tag{8.38}$$

$\sigma(\xi, t)$ exist locally on $[0, T]$, T is maximal time.

Set

$$U(\xi, t) = U_k(\sigma(\xi, t)) \tag{8.39}$$

Claim: $U(\xi, t)$ defined by (8.39), is a solution to the equation in (8.36).

$$\begin{aligned}
\partial_t U &= \frac{DU_k}{D\sigma}\partial_t\sigma = r_k(U_k)\partial_t\sigma \\
\partial_\xi U &= \partial_\xi\sigma\, r_k(U_k) \\
\partial_t U + A(U)\partial_\xi U &= \partial_t\sigma \cdot r_k(U_k) + A(U_k)r_k(U_k)\partial_\xi\sigma \\
&= (\partial_t\sigma + \lambda_k(U_k)\partial_\xi\sigma)r_k(U_k) = 0
\end{aligned}$$

Definition 8.6 The $U_k(\sigma(\xi,t))$ defined by (8.39) is called a simple wave. Recall the previous result on the formation of shocks that (8.38) has a global smooth solution iff

$$\frac{d}{d\xi}\lambda_k(U_k(\sigma_0(\xi))) \geq 0$$

In other words, if $\exists\, \xi_0 \in \mathbb{R}^1$, such that

$$\frac{d}{d\xi}\lambda_k(U_k(\sigma_0(\xi)))|_{\xi=\xi_0} < 0 \tag{8.40}$$

shock must form in finite time

$$\begin{aligned}\frac{d}{d\xi}\lambda_k(U_k(\sigma_0(\xi))) &= \nabla\lambda_k \cdot \frac{dU_k}{d\sigma}\frac{d\sigma_0}{d\xi} \\ &= (\nabla\lambda_k \cdot r_k)\frac{d\sigma_0}{d\xi}\end{aligned}$$

Definition 8.7 (P. D. Lax) The k-th characteristic field is said to be genuinely nonlinear at $u_0 \in \mathcal{D}$ in the direction w, if

$$(\nabla \lambda_k \cdot r_k)(u_0) \neq 0 \tag{8.41}$$

And the k-th field is said to be linearly degenerate if

$$(\nabla \lambda_k \cdot r_k)(u) \equiv 0 \qquad \forall u \in B_\delta(u_0)$$

Proposition 8.6 Assuming that the system in (8.35) is not linearly degenerate in the direction w. Then $\exists$ a k-simple wave which blow-up in finite time, which is determined by

$$\frac{\sigma_0'(\xi)}{1 + (\partial_\xi \lambda_k(u_k(\sigma_0(\xi))))t}$$

Next, blow-up results due to F. John.

$$\begin{cases} \partial_t u + A(u)\,\partial_\xi u = 0 \\ u(\xi, t=0) = u_0(\xi) \end{cases}$$

u_0 has compact support.

Theorem 8.7 (F. John) Assume that

(i) The system in (8.36) is genuinely nonlinear on $B_\delta(\bar{u}_0)$.

(ii) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s > 3$. u_0 has compact support in the sense that

$$u_0 - \bar{u}_0 \in C_0^2(\mathbb{R}^1) \quad \mathrm{supp}\,(u_0 - \bar{u}_0) \subset [a, b]$$

Then there exists a $\theta_0 = \theta_0(\delta, A) > 0$ such that if

$$0 < \theta = (b-a)^2 \, |u_0^{''}|_{L^\infty} \leq \theta_0$$

Then the solution to (8.36) must form shocks in finite time.

Key ideas of the proof:

• Huygen's principle

If $A(u) = A_0$, constant matrix

$$\lambda_1, \cdots, \lambda_n \qquad \text{constant}$$

$$u(x,t) = \sum_{i=1}^{m} \alpha_i \, r_i,$$

- characteristic decomposition of spatial derivatives
- reduced to a Ricatti equation

Step 1: Canonical representation

Let $u(\xi, t)$ be a C^2-smooth solution. Consider the j-th characteristic $\xi = \xi_j(t)$, i.e.

$$\frac{d\xi_j}{dt} = \lambda_j(u(\xi_j(t), t))$$

We denote the differentiation along the j-th characteristic as $\dfrac{d}{dt_j}$, i.e.

$$\frac{d}{dt_j} = \partial_t + \lambda_j \, \partial_\xi$$

Then the system (8.36) can be written as

$$l_j^t(u) \frac{d}{dt_j} u = 0 \qquad j = 1, \cdots, n \tag{8.42}$$

(8.42) is called a canonical representation of (8.36).

Step 2: Characteristic decomposition $\partial_\xi u$

$$\partial_\xi u = \sum_{i=1}^{n} w_i \, r_i(u) \tag{8.43}$$

where $w_i = l_i^t(u)\, \partial_\xi u$.

John's formula

$$\frac{D}{Dt_i} w_i = \sum_{k,l=1}^{n} \gamma_{ikl} \, w_k \, w_l \tag{8.44}$$

$\gamma_{ikl}(u)$ are called interaction coefficients given by

$$\gamma_{ikl} = -\frac{1}{2}(\lambda_k - \lambda_l)\, l_i[r_k, r_l] - (\nabla\lambda_i \cdot r_k)\delta_{il} \tag{8.45}$$

$$[r_k, r_l] = \nabla r_k \cdot r_l - \nabla r_l \cdot r_k$$

Properties of γ_{ikl}

$$\left\{ \begin{array}{llcll} (1) & \gamma_{iii} & = & -\nabla\lambda_i \cdot r_i = -1 & \text{(by normalization)} \\ (2) & \gamma_{ikk} & = & 0 & \text{if } i \neq k \end{array} \right. \tag{8.46}$$

Key idea:

(1) "major" term in (8.44) is $\gamma_{iii}\, w_i^2 = -w_i^2$.

(2) (8.46) implies that no other self-interactions in (8.43), i.e. all the other terms in (8.43) involves $w_j\, w_k$, $j \neq k$ which are the products of waves from different family.

(3) For the initial data with compact support, the approximate Huygen's principle applies, so waves with different speeds eventually separate, thus $w_k\, w_l$ must become smaller for large time, so

$$\frac{d}{dt_i} w_i = \gamma_{iii}\, w_i^2 + O(1)$$

Thus, one can obtain a Ricatti type differential inequality, D_0 blow-up in finite time for w_i. In order to ensure the u still remains $B_\delta(0)$, then one has to show $||\partial_\xi u||_{L^1}$ is bounded.

Remark 8.17 In Theorem 8.6, we require that every characteristic family is genuinely nonlinear, which does not apply to 3×3 gas dynamics equation since for which the entropy wave family is always linearly degenerate.

Theorem 8.7 (JDE, 1979, T. P. Liu) Assume that

(i) The system in (8.36) is strictly hyperbolic.

(ii) Each characteristic field is either genuinely nonlinear or linearly degenerate, $\exists\, N \subset \{1, 2, \cdots, n\}$, such that λ_i is genuinely nonlinear if $i \in N$, λ_j is linearly degenerate if $j \in N^c = \{1, 2, \cdots, n\} \setminus N$.

(iii) Linear waves never generate nonlinear waves, i.e.

$$\gamma_{ikl} = 0 \qquad \text{if} \qquad i \in N \qquad \text{and} \qquad k, l \in N^c \tag{8.47}$$

(iv) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s > 3$, $u - \bar{u}_0 \in C^1(\mathbb{R}^1)$, $\operatorname{supp}(u - \bar{u}_0) \subset [a, b]$.

Then there exists $\theta_0 = \theta_0(\delta, A) > 0$, such that if

$$\theta = (b-a)\,|u_0'|_{L^\infty} \leq \theta_0$$

$$0 < \varepsilon = \max_{i \in N} |w_i(\xi)|_{L^\infty}, \quad w_i(\xi) = l_i^t(u_0(\xi))\,\partial_\xi\, u_0(\xi) \tag{8.48}$$

Then any C^1-solution to problem (8.36) forms shocks in finite time. Furthermore, if $\theta \leq \theta_0$, $\varepsilon = 0$, then smooth solution exists globally.

Remark 8.18 If N^c contains only one element, then (8.47) is satisfied automatically. However, for one-dimensional gas dynamics, only one family (entropy wave family) is linearly degenerate. So Theorem 8.7 indeed applies to 3×3 gas dynamics system.

Remark 8.19 In (8.48), ε measure the strength of the initial nonlinear waves, Theorem 8.7 implies if no nonlinear waves initially, the global smooth solution exists. In particular, if the system is totally linearly degenerate, i.e. $N = \phi$. Then (8.47) is satisfied automatically also. Theorem 8.7 implies global existence of smooth solutions. How about the multi-d case?

Remark 8.20 All the results of F. John has been generalized to the case, the characteristic fields may have inflection points, by Hormander, Da-Tsien Li, etc.

Shock formation for systems endowed with coordinates of Riemann invariants

Definition 8.8 A $c(u)$ is said to be an i-Riemann invariant if

$$\nabla c(u) \cdot r_i(u) \equiv 0 \qquad \forall u \in \mathcal{D} \tag{8.49}$$

Look at (8.49), which is a 1-st order PDE. By the characteristic method, one can find $(n-1)$ i-th Riemann invariants $c_j(u)$, $j = 1, \cdots, n$, $j \neq i$, such that

$$\nabla c_j \cdot r_i = 0$$

and ∇c_j, $j \neq i$, span the orthogonal complement of r_i.

Definition 8.9 The system

$$\partial_t u + A(u)\,\partial_\xi u = 0 \tag{8.50}$$

is said to be endowed with a coordinate system of Riemann invariants, if $\exists\, n$ scalar valued function $c_1(u), \cdots, c_n(u)$ such that $c_j(u)$ is an i-th Riemann invariant for (8.50) for all $j \neq i$, $i, j = 1, \cdots, n$, and $\nabla c_i(u)$, $i = 1, \cdots, n$ are linearly independent.

Proposition 8.7 The functions $(c_1(u), \cdots, c_n(u))$ form a coordinate system of Riemann invariants of (8.50) iff

$$\nabla c_i(u) \cdot r_j(u) = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases} \tag{8.51}$$

Since (8.51) $\Rightarrow c_i(u) // l_i(u)$, therefore

$$(\nabla c_1(u), \cdots, \nabla c_n(u))^T = L(u)$$

Remark 8.21 $\nabla c_i(u)$ must be a left eigenvector of $A(u)$ associated with λ_i.

Recall the canonical form of (8.50)

$$l_i(u)(\partial_t u + \lambda_i \partial_\xi u) = 0, \qquad i = 1, \cdots, n \tag{8.52}$$

Now assume that (8.50) is endowed with a coordinate system of Riemann invariants

$$c(u) = (c_1(u), \cdots, c_n(u))$$

Then

$$l_i(u) = \nabla c_i(u)$$

Then go back to (8.52)

$$\begin{aligned} 0 &= l_i(u)(\partial_t u + \lambda_i(u)\,\partial_\xi u) = \nabla c_i(u)(\partial_t u + \lambda_i(u)\,\partial_\xi u) \\ &= \partial_t\, c_i(u) + \lambda_i(u)\,\partial_\xi\, c_i(u) \end{aligned}$$

$$\partial_t\, c_i + \lambda_i(c)\,\partial_\xi\, c_i = 0 \qquad i = 1, 2, \cdots, n \tag{8.53}$$

Remark 8.22 In the case $n = 2$, this can be done always. However, in general, for $n \geq 3$, the system to determine the invariants is over-determined, thus has no solution.

Proposition 8.8 Assume that (8.50) is endowed with a coordinate of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$. Then

(1) Its canonical form is given by (8.53), which is diagonal system.

(2) For any i, $i = 1, \cdots, n$, $c_i(u)$ is constant along an i-th characteristic associated with any smooth solution.

In particular, for any smooth solution $u(x, t)$

$$||c(u(\cdot, t))||_{L^\infty} = ||c(u_0)||_{L^\infty} \tag{8.54}$$

In the rest of this section, we always assume that (8.50) is endowed with a coordinate of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$, which can be normalized so that

$$\nabla c_i(u) \cdot r_j(u) = \delta_{ij} \tag{8.55}$$

Proposition 8.9 Assume that (8.50) is endowed with a coordinate system of Riemann invariants such that (8.55) hold. Then

$$\text{(i)} \qquad [r_j, r_k] = \nabla r_j \cdot r_k - \nabla r_k \cdot r_j = 0 \qquad \forall j, k \tag{8.56}$$

$$\text{(ii)} \quad r_j^t \nabla^2 c_i r_k = -\nabla c_i \cdot \nabla r_j \, r_k = 0 \qquad i \neq j \neq k \neq i \tag{8.57}$$

$$\text{(iii)} \qquad \frac{\partial g_{jk}}{\partial c_i} = \frac{\partial g_{ji}}{\partial c_k} \qquad i \neq j \neq k \neq i \tag{8.58}$$

$$g_{kj} = \tfrac{1}{\lambda_k - \lambda_j} \tfrac{\partial \lambda_k}{\partial c_j} \tag{8.59}$$

Proof of Proposition 8.9: Recall that $u \mapsto c(u)$ is differomorphism, and

$$\frac{Du}{Dc}\frac{Dc}{Du} = I \Leftrightarrow \frac{Dc}{Du}\frac{Du}{Dc} = I$$

Then it follows from (8.55),

$$\frac{Du}{Dc} = R(u) = (r_1(u), \cdots, r_n(u)), \quad \frac{Dc}{Du} \equiv L(u)$$

i.e.

$$\frac{\partial u}{\partial c_i} = r_i(u), \quad r_i(u) = r_i(u)$$

Thus for any smooth function ϕ,

$$\frac{\partial \phi}{\partial c_i} = \nabla_u \phi \cdot r_i(u) = \nabla_u \phi \cdot r_i(u) \tag{8.60}$$

Step 1:

$$0 = \nabla(\nabla c_i(u) \cdot r_j(u)) r_k = r_j^t \, \nabla^2 \, c_i \, r_k + \nabla c_i \cdot \nabla r_j \, r_k$$

so

$$\nabla c_i \nabla r_j \, r_k = -r_j^t \, \nabla^2 \, c_i \, r_k \qquad \forall \, i, j, k = 1, \cdots, n \tag{8.61}$$

$$\nabla c_i \nabla r_k \, r_j = -r_k^t \, \nabla^2 \, c_i \, r_j \qquad \forall \, i, j, k = 1, \cdots, n$$

$$\nabla c_i [r_j, r_k] = 0 \qquad \Leftrightarrow \qquad [r_j, r_k] = 0$$

since it is true for all i, $\Rightarrow$

By (8.60), this is equivalently

$$\frac{\partial r_j}{\partial c_k} = \frac{\partial r_k}{\partial c_j}$$

Step 2:

$$\begin{aligned}
Ar_j &= \lambda_j \, r_j \\
\nabla(Ar_j)r_k &= \nabla(\lambda_j r_j)r_k = \nabla\lambda_j r_k r_j + \lambda_j \nabla r_j r_k \\
Ar_k &= \lambda_k \, r_k \\
\nabla(Ar_k)r_j &= \nabla(\lambda_k \, r_k)r_j = \nabla\lambda_k r_j r_k + \lambda_k \nabla_k r_j \\
r_j^t \, \nabla A r_k + A\nabla r_j \, r_k &= \nabla\lambda_j \, r_k \, r_j + \lambda_j \nabla r_j \, r_k \\
r_k^t \, \nabla A r_j + A\nabla r_k \, r_j &= \nabla\lambda_k \, r_j \, r_k + \lambda_k \nabla r_k \, r_j
\end{aligned}$$

Since $A = \nabla F$, so ∇A is symmetric. Taking the difference, we have

$$A[r_j, r_k] = (\nabla\lambda_j\, r_k)r_j - (\nabla\lambda_k\, r_j)r_k + \lambda_j\, \nabla r_j\, r_k - \lambda_k\, \nabla r_k\, r_j$$

$$\begin{aligned} (\nabla\lambda_j\, r_k)r_j - (\nabla\lambda_k\, r_j)r_k &= \lambda_k\, \nabla r_k\, r_j - \lambda_j\, \nabla r_j\, r_k \\ &= (\lambda_k - \lambda_j)\nabla r_j\, r_k \end{aligned} \tag{8.62}$$

This implies that $\nabla r_j\, r_k$ is a linear combination of r_j and r_k. Now for $i \neq j$, $i \neq k$, $j \neq k$

$$\begin{aligned} \nabla c_i\, \nabla r_j\, r_k &= \frac{\nabla\lambda_j\, r_k}{\lambda_k - \lambda_j}\nabla c_i\, r_j - \frac{\nabla\lambda_k\, r_j}{\lambda_k - \lambda_j}\nabla c_i\, r_k \\ &= 0 \end{aligned} \tag{8.63}$$

Then (8.57) follows from (8.61) and (8.63).

Step 3: By (8.62),

$$\frac{\partial r_j}{\partial c_k} = \frac{\frac{\partial \lambda_j}{\partial c_k}}{\lambda_k - \lambda_j} r_j - \frac{\frac{\partial \lambda_k}{\partial c_j}}{\lambda_k - \lambda_j} r_k$$

i.e.

$$-\frac{\partial r_j}{\partial c_k} = g_{jk}\, r_j + g_{kj}\, r_k, \qquad j, k = 1, \cdots, n, \quad j \neq k \qquad (8.64)$$

Differentiate the equality with respect to c_i,

$$-\frac{\partial^2 r_j}{\partial c_k\, \partial c_i} = \frac{\partial g_{jk}}{\partial c_i} r_j + g_{jk} \frac{\partial r_j}{\partial c_i} + g_{kj} \frac{\partial r_k}{\partial c_i} + \frac{\partial g_{kj}}{\partial c_i} r_k$$

Substitute (8.64) into this formula,

$$-\frac{\partial^2 r_j}{\partial c_k\,\partial c_i} = \frac{\partial g_{jk}}{\partial c_i} r_j - g_{jk}(g_{ji}\, r_j + g_{ij}\, r_i) - g_{kj}(g_{ki}\, r_k + g_{ik}\, r_i) + \frac{\partial g_{kj}}{\partial c_i} r_k$$

By the symmetry of i and k,

$$-\frac{\partial^2 r_j}{\partial c_i\,\partial c_k} = \frac{\partial g_{ji}}{\partial c_k} r_j - g_{ji}(g_{jk}\, r_j + g_{kj}\, r_k) - g_{ij}(g_{ik}\, r_i + g_{ki}\, r_k) + \frac{\partial g_{ij}}{\partial c_k} r_i$$

This implies

$$\left(\frac{\partial g_{jk}}{\partial c_i} - \frac{\partial g_{ji}}{\partial c_k}\right) r_j + r_k(\qquad\qquad\qquad\qquad) + r_i(\qquad\qquad\qquad\qquad) = 0$$

so

$$\frac{\partial g_{jk}}{\partial c_i} = \frac{\partial g_{ji}}{\partial c_k}$$

Theorem 8.8 Assume that

(i) (8.50) is endowed with a coordinate system of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$.

(ii) (8.50) is strictly hyperbolic.

(iii) $\exists\, i \in \{1, \cdots, n\}$ such that the i-th family is genuinely nonlinear

$$\nabla\lambda_i\, r_i \neq 0 \quad \left(\frac{\partial\lambda_i}{\partial c_i} \neq 0\right)$$

(iv) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s \geq 3$ and $\exists\, \xi_0 \in \mathbb{R}^1$ such that

$$\frac{d\, c_i(u_0(\xi_0))}{d\xi}\frac{\partial\lambda_i}{\partial c_i} < 0, \quad \frac{\partial\lambda_i}{\partial c_i} = \nabla\lambda_i(u_0(\xi_0)) \cdot r_i(u_0(\xi_0)) \quad (8.65)$$

Then smooth solution forms a shock in finite time.

Proof of Theorem 8.8:

Step 1: By (8.54) in Proposition 8.8, $||c(u(\cdot,t))||_{L^\infty} = ||c(u_0)||_{L^\infty}$, so there are no shell singularities.

Step 2: To estimate $\partial_\xi u$. Set

$$\partial_\xi u = \sum_{i=1}^{n} w_i\, r_i, \qquad w_i = l_i \cdot \partial_\xi u = \nabla c_i(u)\, \partial_\xi u \tag{8.66}$$

so

$$w_i = \partial_\xi\, c_i \tag{8.67}$$

$$\frac{d}{dt} w_i = \partial_t\, w_i + \lambda_i\, \partial_\xi\, w_i = \sum \gamma_{ijk}\, w_k\, w_j \tag{8.68}$$

and

$$\begin{aligned}\gamma_{ijk} &= -\frac{1}{2}(\lambda_j - \lambda_k) l_i\,[r_j, r_k] - (\nabla\lambda_i \cdot r_j)\delta_{ik} \\ &= -\frac{\partial\lambda_i}{\partial c_j}\delta_{ik}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt} w_i &= \sum_{j,k}\left(-\frac{\partial\lambda_i}{\partial c_j}\delta_{ik}\right) w_k\, w_j \\ &= \sum_{j}\left(-\frac{\partial\lambda_i}{\partial c_j} w_i\, w_j\right) \\ &= -\frac{\partial\lambda_i}{\partial c_i} w_i^2 - \left(\sum_{j\neq i}\frac{\partial\lambda_i}{\partial c_j} w_j\right) w_i\end{aligned} \tag{8.69}$$

Step 3: Find an integration factor for (8.69)

$$\begin{aligned}
&\frac{d}{dt}\Phi(u) \\
= \;& \Phi'(u)\frac{du}{dt} \qquad\qquad\qquad (\text{In fact, } \Phi'(u) = \nabla\Phi(u)) \\
= \;& \Phi'(u)\left(\frac{\partial u}{\partial t} + \lambda_i\,\partial_\xi u\right)
\end{aligned}$$

$$\partial_t u = -A\partial_\xi u = -A\sum_j w_j\,r_j = -\sum_j w_j\,\lambda_j\,r_j$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\Phi(u) &= \Phi'(u)\left(-\sum_j \lambda_j\,w_j\,r_j + \lambda_i\sum_j w_j\,r_j\right) \\
&= \Phi'(u)\sum_{j\neq i}(\lambda_i - \lambda_j)w_j\,r_j
\end{aligned}$$

Thus for any smooth function $\Phi(u)$,

$$
\begin{aligned}
\frac{d}{dt}\left(e^{\Phi(u)}w_i\right) &= \frac{d}{dt}e^{\Phi(u)}w_i + e^{\Phi(u)}\frac{d}{dt}w_i \\
&= e^{\Phi(u)}\frac{d}{dt}w_i + e^{\Phi(u)}\Phi'(u)\sum_{j\neq i}(\lambda_i - \lambda_j)w_j\, r_j\, w_i \\
&= e^{\Phi(u)}\left\{-\sum \frac{\partial \lambda_i}{\partial c_j} w_i\, w_j + \nabla\Phi(u)\sum_{j\neq i}(\lambda_i - \lambda_j)w_i\, w_j\, r_j\right\} \\
&= e^{\Phi(u)}\left\{-\frac{\partial \lambda_i}{\partial c_i}w_i^2 - \sum_{j\neq i}\left(\frac{\partial \lambda_i}{\partial c_j} - \nabla\Phi(u) r_j(\lambda_i - \lambda_j)\right) w_i\, w_j\right\} \\
&= e^{\Phi(u)}\left\{-\frac{\partial \lambda_i}{\partial c_i}w_i^2 - \sum_{j\neq i}\left(\frac{\partial \lambda_i}{\partial c_j} - \frac{\partial \Phi}{\partial c_j}(\lambda_i - \lambda_j)\right) w_i\, w_j\right\}
\end{aligned}
$$

Claim: One can choose an integral factor $\Phi(u)$ such that

$$\frac{\partial \Phi}{\partial c_j} = \frac{\frac{\partial \lambda_i}{\partial c_j}}{\lambda_i - \lambda_j} \qquad j \neq i \tag{8.70}$$

Assume that the claim (8.70) holds

$$\frac{d}{dt}\left(e^{\Phi(u)} w_i\right) = -\frac{\partial \lambda_i}{\partial c_i} e^{-\Phi(u)} \left(e^{\Phi(u)} w_i\right)^2$$

Claim is followed from (8.58) and (8.59).