Section 6. Asymptotic Behavior of Weak Solutions for System of Conservation Laws

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \qquad u \in \mathbb{R}^n & (6.1) \\ u(x, t = 0) = u_0(x) & (6.2) \end{cases}$$

$$\lim_{x \to +\infty} u_0(x) = u_r, \qquad \lim_{x \to -\infty} u_0(x) = u_l$$

$u_l,\ u_r \in \mathbb{R}^n$ are two constant states.

A1 (6.1) is strictly hyperbolic.

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \quad u \in \mathbb{R}^n$$

right eigenvector $\quad \gamma_1(u),\ \gamma_2(u)\ \cdots\ \gamma_n(u)$

left eigenvector $\quad l_1(u),\ l_2(u)\ \cdots\ l_n(u)$

A2 Each characteristic field is either genuinely nonlinear or linearly degenerate.

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x, t = 0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \end{cases}$$

$u_R(x, t)$, $\exists\, u_0 = u_l, u_1, u_2, \cdots, u_n = u_r$ such that $u_i = T_i(u_{i-1})$, here $T_i(u)$ is the i-th wave curve through the base point u.

Let $U(x, t)$ be the unique viscosity solution to the Cauchy problem (6.1) - (6.2). Then one can regard $U(x, t)$ as a limit of approximate solutions constructed by Glimm's method.

Goal: What will be large time asymptotic behavior of $U(x, t)$, as $t \to +\infty$?

If $n = 1$, f is convex, then this problem is well understood.

Case 1: $u_l = u_r$, $u_R(x, t) = u_l = u_r$, then $u(x, t) - u_r \to 0$ as $t \to \infty$, with the decay rate $\left(\frac{1}{\sqrt{t}}\right)$.

Case 2: $u_l > u_r$, $U(x, t) \to u_R$ with a phase shift (shock).

Case 3: $u_l < u_r$, $U(x, t) \to u_R$ (rarefaction wave).

Conclusion ($n = 1$). The large time behavior of $U(x,t)$ is determined completely by the far fields of the initial data (u_l, u_r), in other words, the Riemann solution is "stable".

Question: What happens for $n > 1$?
Let $U(x,t)$ be a Glimm solution. By the strict hyperbolicity,

$$\lambda_1(U(x,t)) < \lambda_2(U(x,t)) < \cdots < \lambda_n(U(x,t))$$

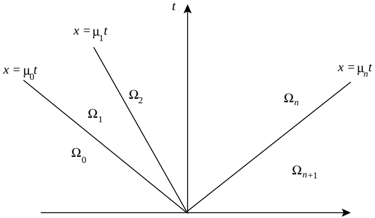
$\exists\, \delta > 0$, μ_i, $i = 0, 1, \cdots, n$ such that

$$\mu_0 + \delta \le \min_{(x,t)} \ \lambda_1(U(x,t))$$

$$\max_{(x,t)} \ \lambda_i(U(x,t)) + \delta \le \mu_i \le \min_{(x,t)} \ \lambda_{i+1}(U(x,t)) - \delta, \qquad i = 1, \cdots, n-1$$

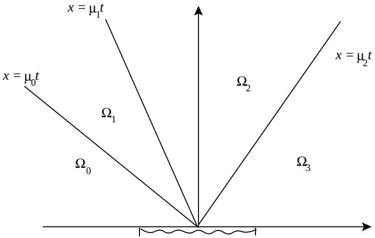
$$\max_{(x,t)} \ \lambda_n(U(x,t)) + \delta \le \mu_n$$

Primary Region: Ω_i, $i = 0, 1, \cdots, n+1$, is defined as



$$\begin{aligned}
\Omega_0 &= \{(x,t);\ x < \mu_0\, t\} \\
\Omega_1 &= \{(x,t);\ \mu_{i-1}\, t < x < \mu_i\, t\} \qquad i = 1, \cdots, n \\
\Omega_{n+1} &= \{(x,t);\ x > \mu_n\, t\}
\end{aligned}$$

Example: $n = 2$.



Theorem 6.1 (Asymptotic behavior toward Riemann Solutions)

Let $U(x,t)$ be the viscosity solution to (6.1) and (6.2) with small initial total variation. Let $u_R(x,t)$ be the corresponding Riemann solution w, γ, t, (u_l, u_r) solved by elementary waves (u_{i-1}, u_i) as described before. Then

(1) $U(x,t) \to u_i \;\text{ as }\; t \to +\infty \;\text{ as }\; \dfrac{x}{t} = \mu_i.$

(2) If $\nabla\lambda_i \cdot \gamma_i > 0$, $i = \alpha_1, \cdots, \alpha_p$.
(u_{i-1}, u_i) is i-rarefaction wave, i.e., $\lambda_i(u_{i-1}) \leq \lambda_i(u_i)$. Then the amount of i-shock wave in Ω_i approach zero as $t \to \infty$ and $U(x,t)$ approaches the centered rarefaction waves (u_{i-1}, u_i).

(3) If $\nabla\lambda_i \cdot \gamma_i > 0$, $i = \alpha_1, \cdots, \alpha_p$ and (u_{i-1}, u_i) is an i-shock, i.e., $\lambda_i(u_{i-1}) > \lambda_i(u_i)$. Then in Ω_i, the solution $U(x,t)$ approaches (u_{i-1}, u_i) both in strength and in shock speed, furthermore, the total variation of $U(x,t)$ in Ω_i away from the shock approach zero.

(4) If $\nabla\lambda_i \cdot \gamma_i \equiv 0$, $i = \beta_1, \cdots, \beta_{n-p}$. In this case, (u_{i-1}, u_i) is an i-contact discontinuity, and $\lambda_i(u_{i-1}) = \lambda_i(u_i)$. Then

$$\lambda_i(U(x,t)) \to \lambda_i(U_i) = \lambda_i(U_{i-1}), \quad (x,t) \in \Omega_i, \quad t \to +\infty.$$

The distance between $\{U(x,t),\ (x,t) \in \Omega_i\}$ and $T_i\, u_{i-1} = T_i\, u_i$ approach zero as $t \to +\infty$.

Main Idea of the Proof:

(1) Nonlinearity introduces dissipation: expansion waves cancels compressive waves (due to entropy condition).

(2) Decoupling of waves $\longleftrightarrow$ nonlinear superposition principle $\Longleftrightarrow$ when $t \gg 1$, only i-wave dominates on Ω_i.

Theorem 6.2

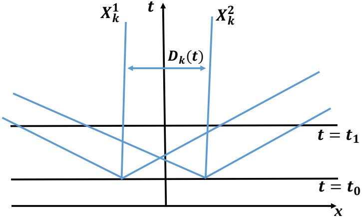
Let $X_k^i (i = 1, 2;\ 1 \leq k \leq n)$ be two generalized k-characteristic issued from two points on $t = t_0$, with $X_k^1 \leq X_k^2$, let $t_1 (\geq t_0)$ be any time after which X_k^j does not intersect $X_i^{j'}$ for $i \neq k$. Denote by $D_k(t)$ the distance between X_k^1 and X_k^2, i.e.

$$D_k(t) = X_k^2(t) - X_k^1(t)$$

$\mathbb{X}_k^+(t)$: amount of k-rarefaction wave between X_k^1 and X_k^2.
$\mathbb{X}_k^-(t)$: amount of k-shock between X_k^1 and X_k^2 (does not include X_k^i, $i = 1, 2$). Then for $t > t_1$,

$$\mathbb{X}_k^+(t) \leq \frac{D_k(t)}{t - t_1} + O(1)[Q_k(t_0, t) + h_k(t_0, t)]$$

where $Q_k(t_0, t)$ is amount of wave interaction between t_0 and t and X_k^1 and X_k^2, $h_k(t_0, t)$ is the total amount of i-waves crossing X_k^1 (X_k^2) for all $i > k$ ($i < k$) between t_0 and t.



Proof of Theorem 6.2

Step 1: Approximation conservation laws

Let Λ be a region bounded by either generalized characteristic or space like curves.

$$L_i^{\pm}(\Lambda) = E_i^{\pm}(\Lambda) \mp c_i(\Lambda) + O(1)\, Q(\Lambda)$$

$L_i(\Lambda)$	:	i-waves leaving Λ.
$E_i(\Lambda)$	:	i-waves entering Λ.
$c(\Lambda)$	:	i-wave cancellation in Λ.
$Q(\Lambda)$	:	interaction happening in Λ.

Step 2: Expansion of rarefaction waves

Recall that a generalized characteristic curve is piecewise Lipschitz continuous which is either a genuine characteristic or a shock. $\tilde{\mathbb{X}}_k(t)$ is the total amount of j-waves ($j \neq k$) between $X_k^1(t)$ and $X_k^2(t)$.

Denote $u_k^{\pm i}(t) = U(X_k^i(t)\pm, t)$, then

$$\begin{aligned} \dot{D}_k(t) = \frac{d}{dt} D_k(t) &= \frac{d}{dt} (X_k^2(t)) - \frac{d}{dt} X_k^1(t) \\ &= \lambda_k(u_k^{+2}(t), u_k^{-2}(t)) - \lambda_k(U_k^{+1}(t), U_k^{-1}(t)) \end{aligned}$$

Fact 1: $\exists\,\theta(t) \in (0,1)$ such that

$$\dot{D}_k(t) = \theta(t)(\lambda_k^{-2}(t) - \lambda_k^{+1}(t)) + (1-\theta(t))(\lambda_k^{+2}(t) - \lambda_k^{-1}(t))$$

where $\lambda_k^{\pm i}(t) = \lambda_k(U(X_k^i(t)\pm, t))$ (due to entropy condition).

In fact, $\exists\,\theta \in (0,1)$ such that

(1) $\dot{D}_k(t) \leq \theta(\lambda_k^{-2}(t) - \lambda_k^{+1}(t)) + (1-\theta)\,(\lambda_k^{+2}(t) - \lambda_k^{-1}(t))$.

(2) $\dot{D}_k(t) = (\lambda_k^{-2}(t) - \lambda_k^{+1}(t)) + (1-\theta(t))$ (strength X_k^1 + strength X_k^2).

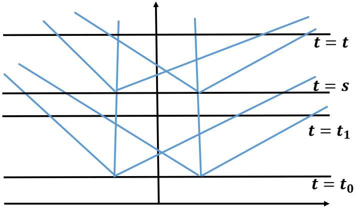
Indeed,

$$\dot{D}_k(t) = \theta(t)\,(\lambda_k^{-2}(t) - \lambda_k^{+1}(t)) + (1-\theta(t))$$
$$(\lambda_k^{-2}(t) - \lambda_k^{-2}(t) + \lambda_k^{-2}(t) - \lambda_k^{+1}(t) + \lambda_k^{+1}(t) - \lambda_k^{-1}(t)).$$

Fact 2:

(1) $\exists\, t_1 > t_0$, such that all X_k^j $(k \neq k^1)$ crosses $X_{k'}^{j'}$ before t_1.

(2) $\forall\, t > t_1$, $\exists\, s$ such that $t - s = O(1)\, D_k(t)$ such that X_{k-1}^2 crosses X_k^1 and X_{k+1}^1 crosses X_k^2 before t.



Step 2.1: Estimate $\tilde{X}_k(t)$ (total amount of i-waves ($i \neq k$) crossing $(X_k^1(t), X_k^2(t))$).

If $i < k$, by applying approximation conservation laws.

$$\tilde{\mathbb{X}}_k^i(t) \leq O(1) \int_s^t \, d(h_k(t_0, \tau) + Q_k(t_0, \tau))$$

for $i > k$, similar estimate also holds, so,

$$\tilde{\mathbb{X}}_k(t) \leq O(1) \int_s^t \, d(h_k(t_0, \tau) + Q_k(t_0, \tau))$$

Step 2.2:

$$
\begin{aligned}
& \lambda_k^{-2}(t) - \lambda_k^{+1}(t) \\
= \;& \mathbb{X}_k^+(t) + \mathbb{X}_k^-(t) + O(1)\ \tilde{\mathbb{X}}_k(t) \\
= \;& \mathbb{X}_k^+(t) + \mathbb{X}_k^-(t) + O(1) \int_s^t d(Q_k(t_0,\tau) + h_k(t_0,\tau))
\end{aligned}
$$

Then

$$
\begin{aligned}
\dot{D}_k(t) &= \lambda_k^{-2}(t) - \lambda_k^{-1}(t) + (1-\theta(t))\ (\text{str } X_k^2(t) + \text{str } X_k^1(t)) \\
&= \mathbb{X}_k^+(t) + \mathbb{X}_k^-(t) + O(1) \int_s^t d(Q_k(t_0,\tau) + h_k(t_0,\tau)) \\
&\quad + (1-\theta(t))\ (\text{str } X_k^2(t) + \text{str } X_k^1(t))
\end{aligned}
$$

Integrate with respect to t from t_1 to t,

$$\begin{aligned} & D_k(t) - D_k(t_1) \\ = \; & \int_{t_1}^{t} (\mathbb{X}_k^+(\tau) + \mathbb{X}_k^-(\tau)) d\tau + \int_{t_1}^{t} (1-\theta(\tau)) \; (\text{str } X_k^2(\tau) + \text{str } X_k^1(\tau)) d\tau \\ & + O(1) \; \int_{t_1}^{t} \int_{s}^{\tau} \; d(Q_k(t_0,\xi) + h_k(t_0,\xi)) d\tau \\ = \; & \int_{t_1}^{t} (\mathbb{X}_k^+(\tau) + \mathbb{X}_k^-(\tau) + (1-\theta(\tau)) \; (\text{str } X_k^2(\tau) + \text{str } X_k^1(\tau)) d\tau \\ & + O(1) \int_{t_1}^{t} (\tau - s) \; d \; (Q_k(t_0,\tau) + h_k(t_0,\tau)) \end{aligned}$$

Recall the approximate conservation law

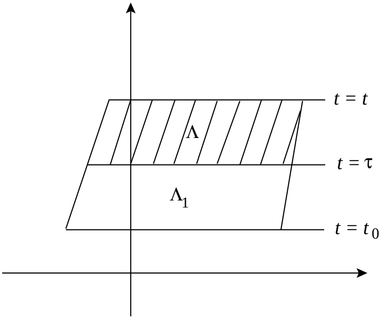
$$L_i^{\pm}(\Lambda) = E_i^{\pm}(\Lambda) \mp c_i(\Lambda) + O(1) \; Q(\Lambda) \tag{6.3}$$

$$\left(\text{for } \alpha, \beta, \; c(\alpha,\beta) = \frac{1}{2}(|\alpha| + |\beta| - |\alpha + \beta|)\right)$$

So if we apply (6.3) to Λ,

$$\begin{aligned}\mathbb{X}_k^+(t) &= \mathbb{X}_k^+(\tau) - c_k(\Lambda) + O(1)\,Q_k(\tau,t)\\ &\leq \mathbb{X}_k^+(\tau) + O(1)\,Q_k(\tau,t)\end{aligned}$$

$$\text{i.e.}\quad \mathbb{X}_k^+(\tau) \geq \mathbb{X}_k^+(t) + O(1)\,Q_k(\tau,t)$$



Similarly, applying (6.3) to Λ_1,

$$\begin{aligned} \mathbb{X}_k^-(\tau) &= \mathbb{X}_k^-(t_0) + c_k(\Lambda_1) + O(1)\, Q_k(t_0,\tau) \\ &\geq \mathbb{X}_k^-(t_0) + O(1)\, Q_k(t_0,\tau) \end{aligned}$$

Therefore,

$$\begin{aligned} D_k(t) \;\geq\;& D_k(t_1) + \mathbb{X}_k^+(t)(t-t_1) \\ &+O(1)\int_{t_1}^{t} Q_k(\tau,t)d\tau + \mathbb{X}_k^-(t_0)(t-t_1) \\ &+O(1)\int_{t_1}^{t} Q_k(t_0,\tau)d\tau \\ &+O(1)\int_{t_1}^{t} (1-\theta(\tau))\,(\text{str } X_2(\tau) + \text{str } X_1(\tau))d\tau \\ &+\; O(1)\int_{t_1}^{t} D_k(\tau)\; d\; (Q_k(t_0,\tau) + h_k(t_0,\tau)) \end{aligned}$$

so,

$$
\begin{aligned}
& \mathbb{X}_k^+(t) \\
\leq \quad & \frac{D_k(t)}{t-t_1} + \left(-\mathbb{X}_k^-(t_0) + \frac{(-1)}{t-t_1} \int_{t_1}^{t} (1-\theta(\tau)) \right) (\text{str } X_2(t) + \text{str } X_1(\tau)) d\tau \\
& + O(1)\, Q_k(t_0, t) + O(1) \frac{1}{t-t_1} \int_{t_1}^{t} D_k(\tau)\ d\ (Q_k(t_0,\tau) + h_k(t_0,\tau))
\end{aligned}
$$

Immediately, we obtain that

$$
\mathbb{X}_k^+(t) \leq \frac{D_k(t)}{t-t_1} + O(1)[-\mathbb{X}_k^-(t_0) - \max \text{ str } X_k(\tau) + Q_k(t,t_0) + h_k(t,t_0)]
$$

This is true for any two characteristic, just do this procedure for the increasing variation part.

Next, we turn to the Proof of Theorem 6.1.

Lemma 6.1

$\exists\, \delta_0 > 0$, such that if $T.V.\, u_0 \leq \delta_0$, then

(1) $T.V.\, u(\cdot, t) \leq c_0\, \delta_0 \qquad \forall\, t > 0.$

(2) $Q(0, t) \leq c_1 \qquad \forall\, t > 0.$

here $Q(t_1, t_2)$ is the total amount of wave interaction taken place between t_1 and t_2.

Proof of Lemma 6.1: Since $u(x,t)$ is a solution generated by Glimm's scheme, so (1) is true for the Glimm approximate solution, thus it is true for its limit.

To see (2), we consider any J-curve J and its immediate successor,

$$\begin{aligned} Q(J') - Q(J) &\le -D(\Delta) + O(1)\, D(\Delta)\, L(J) \\ &\le -\frac{1}{2}\, D(\Delta) \qquad \text{(by (1)}, L(J) \text{ is sufficiently small.)} \end{aligned}$$

Now, we consider a region Λ whose domain of dependence contains a mesh curve J, and Λ consists of all diamonds. Then summing the above inequality up, we can get

$$Q(\Lambda) = \sum_{\Delta \in \Lambda} D(\Delta) \le 2Q(J) \le c_1.$$

Take limit to Glimm's approximate solution

$$Q(\Lambda) \leq c_1 \Longrightarrow Q(0, t) \leq c_1.$$

Lemma 6.2

$\forall\, \varepsilon > 0$, $\exists\, t_0 = t_0(\varepsilon)$ and $M = M(\varepsilon)$ such that

(1) $Q(t_0, t) < \varepsilon \qquad \forall\, t > t_0.$

(2) $T.V._{\{|x| \geq M\}}\ u(\cdot, t_0) < \varepsilon.$

Proof of Lemma 6.2: These follow from Lemma 6.1.

Lemma 6.3

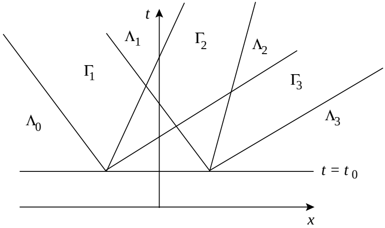
Let Γ_i be the region between X_i^1 and X_i^2, Λ_0 be the region left of X_1^1, Λ_i be the region between X_1^2 and X_{i+1}^1, $i = 1, \cdots, n-1$, and Λ_n be the right of X_n^2. Then for any $t \geq t_1$.

(1) $\overset{T.V.}{\Lambda_i} \; u(\cdot, t) = O(\varepsilon)$.

(2) The amount of j-waves outside of Γ_j at time t is $O(\varepsilon)$.

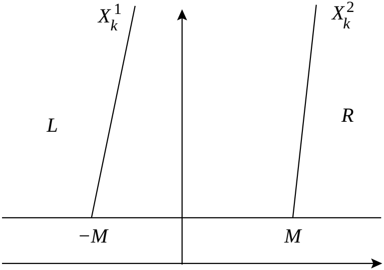
(3) $\overset{\text{osc}}{\Lambda_i} \; u(\cdot, \cdot) = O(1)\,\varepsilon$.

(4) $\mathbb{X}_j^+(t) \le \dfrac{D_j(t)}{t - t_1} + O(1)\,\varepsilon$.



Proof of Lemma 6.3: We start with (2). Applying approximate conservation law to the Region R (which is the right of X_k^2 for $i \leq k$), one can get

$$L_i^{\pm}(R) \leq E_i^{\pm}(R) + O(1)\, Q(R).$$



Therefore, the total amount of i-wave ($i < k$) crossing $X_k^2 = O(1)\,\varepsilon$, the total amount of k-waves on the right of $X_k^2 = O(1)\,\varepsilon$. Similarly, one can apply the approximate conservation law to L (left of X_k^1) for $i \geq k$, then the total amount of i-wave ($i > k$) crossing $X_k^1 = O(1)\,\varepsilon$, the total amount of k-wave on the left of $X_k^1 = O(1)\,\varepsilon$.

Thus (2) is true, and

$$h_k(t_0, t) = O(1)\,\varepsilon.$$

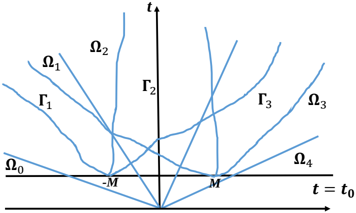
Immediately,

$$\mathbb{X}_k^+(t) \leq \frac{D_k(t)}{t - t_1} + O(1)\,\varepsilon,$$

(1) and (3) are consequence of (2).

Lemma 6.4

The total amount of i-waves in the region Ω_j, $j \neq i$, $j = 0, 1, \cdots, n, n+1$ at time t approaches to 0 as $t \to +\infty$.



Proof of Lemma 6.4: By the strict hyperbolicity, for large t, one has

$$\Gamma_i \subset \Omega_i,$$

so the conclusions follows from Lemma 6.3.

Lemma 6.5 (Emergency of contact waves)

Assume that $\nabla\lambda_i \cdot \gamma_i \equiv 0$, $\forall\, i = \beta_1, \cdots, \beta_{n-p}$. Then $\forall\ (x_k, t_k) \in \Lambda_k,\ k = i-1, i,\ i = \beta_i, \cdots, \beta_{n-p}$.

(1) $\lambda_i(u(x_i, t_i)) = \lambda_i(u(x_{i-1}, t_{i-1})) + O(\varepsilon)$.

(2) $U(x_i, t_i) \in T_i(u(x_{i-1}, t_{i-1})) + O(\varepsilon)$.

Proof of Lemma 6.5: By (3) of Lemma 6.3, without loss of generality, $t_{i-1} = t_i$. Since $\lambda_i(u(\cdot, t_i))$ changes only when it crosses j-waves for $j \neq i$ which is of order $O(\varepsilon)$.

Lemma 6.6 (Emergency of shock wave)

Suppose

$$\nabla\lambda_i(u)\cdot\gamma_i(u) > 0, \qquad i=\alpha_1,\cdots,\alpha_p\ .$$

Then $\exists\, k_0$ such that if

$$\lambda_i(u(x_i,t_i)) \le \lambda_i(u(x_{i-1},t_{i-1})) - k_0\,\varepsilon, \qquad (x_i,t_i)\in\Lambda_i\ .$$

Then for sufficiently large t

(1) $\mathbb{X}_i^+(t) = O(\varepsilon)$.

(2) X_i^1 and X_i^2 collide to form an i-shock with strength

$$\lambda_i(u(x_i,t_i)) - \lambda_i(u(x_{i-1},t_{i-1})) + O(\varepsilon).$$

Proof of Lemma 6.6: Recall that

$$
\begin{aligned}
\lambda_i^{-2}(t) - \lambda_i^{+1}(t) &= \mathbb{X}_i^{+}(t) + \mathbb{X}_i^{-}(t) + O(1)\ \tilde{\mathbb{X}}_i(t) \\
&= \mathbb{X}_i^{+}(t) + \mathbb{X}_i^{-}(t) + O(1)\,\varepsilon
\end{aligned}
$$

$$
\lambda_i^{+2}(t) - \lambda_i^{-1}(t) = \lambda_i(U(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1})) + O(\varepsilon)
$$

$$
\begin{aligned}
\dot{D}(t) \le\ & \theta(\lambda_i^{-2}(t) - \lambda_i^{+1}(t)) + (1-\theta)(\lambda_i^{+2} - \lambda_i^{-1}(t)) \qquad (0 < \theta < 1) \\
=\ & \theta(\mathbb{X}_i^{+}(t) + \mathbb{X}_i^{-}(t)) + (1-\theta)(\lambda(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1}))) + O(\varepsilon) \\
\le\ & \theta\,\mathbb{X}_i^{+}(t) + (1-\theta)(\lambda_i(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1}))) + O(\varepsilon) \\
\le\ & \theta\ \frac{D_i(t)}{t - t_1} + (1-\theta)(\lambda_i(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1}))) + O(\varepsilon)
\end{aligned}
$$

Set

$$H_i(t) = D_i(t) - [\lambda_i(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1}))](t - t_1).$$

Then

$$\dot{H}_i(t) \leq \theta \ \frac{H(t)}{t - t_1} + O(\varepsilon).$$

Solving this differential inequality

$$H_i(t) \leq (t - t_1)^\theta \ H_i(t_1 + 1) + O(\varepsilon)(t - t_1).$$

Thus,

$$\begin{aligned} D_i(t) \quad &\leq \quad (\lambda_i(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1})))(t - t_1) \\ &\quad + O(1)\ (t - t_1)^\theta + O(\varepsilon)\ (t - t_1) \qquad (\bigstar) \\ &= \quad [(\lambda_i(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1}))) \\ &\quad + O(1)\,\varepsilon](t - t_1) + O(1)\ (t - t_1)^\theta \end{aligned}$$

Choose k_0 sufficiently large, then

$$D_i(t) < 0 \qquad \text{for} \quad t \gg 1,$$

so the conclusions follows.

Lemma 6.7 (Emergency of rarefaction waves)

Assume that $\lambda_i(U(x_i, t_i)) - \lambda_i(U(x_{i-1}, t_{i-1})) \geq -O(1)\,\varepsilon$ for some uniform constant $O(1) \geq 0$. Then

(1) $|\mathbb{X}_i^-(t)| + |\text{str } X_i^j(t)| = O(1)\,\varepsilon$.

(2) $U(x_i, t_i) \in R_i^+(u(x_{i-1}, t_{i-1})) + O(1)\,\varepsilon$.

Proof of Lemma 6.7: By Lemma 6.3,

$$\begin{array}{lrcl} & \mathbb{X}_i^+(t) & \leq & \dfrac{D_i(t)}{t-t_1} + O(1)\,\varepsilon \\ \text{(By } (\bigstar)) & & \leq & \lambda_i(u(x_i,t_i)) - \lambda_i(u(x_{i-1},t_{i-1})) \\ & & & +O(1)(t-t_1)^{\theta-1} + O(1)\,\varepsilon \\ & & \leq & [\lambda_i(u(x_i,t_i)) - \lambda_i(u(x_{i-1},t_{i-1}))] \\ & & & +O(1)\,\varepsilon \qquad \text{for large } t. \end{array}$$

On the other hand,

$$\begin{array}{rl} & \lambda_i(u(x_i,t_i)) - \lambda_i(u(x_{i-1},t_{i-1})) \\ = & \lambda_i^{+2}(t) - \lambda_i^{-1}(t) + O(1)\,\varepsilon \\ = & \mathbb{X}_i^+(t) + \mathbb{X}_i^-(t) + O(1)\,\tilde{\mathbb{X}}_i(t) + \text{str}\,X_i + O(1)\,\varepsilon \end{array}$$

so,

$$\begin{aligned} & \mathbb{X}_i^+(t) - [\lambda_i(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1}))] \\ = \ & |\mathbb{X}_i^-(t)| + |\text{str } X_i| - O(1)\,\varepsilon \\ \geq \ & -O(1)\,\varepsilon \end{aligned}$$

Thus,

$$\mathbb{X}_i^+(t) = \lambda_i(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1})) + O(1)\,\varepsilon,$$

so,

$$|\mathbb{X}_i^-(t)| + |\text{str } X_i(t)| = O(1)\,\varepsilon.$$

We need to relate (u_{i-1}, u_i) in $u_R(x, t)$ to $(u(x_{i-1}, t_{i-1}),\ u(x_i, t_i))$.

Lemma 6.8 (Comparison with the Riemann solution)

Let (u_{i-1}, u_i) be the i-th wave in the Riemann solution $u_R(x,t) = u\left(\frac{x}{t}\right)$,

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,0) = \begin{cases} u_l & x < 0 \\ u_R & x > 0 \end{cases} \end{cases}$$

Then

$$|u(x_i,t_i) - u_i| = O(1)\,\varepsilon, \qquad \forall\,(x_i,t_i) \in \Lambda_i\ .$$

Proof of Lemma 6.8 It follows from Lemmas 6.5, 6.6, 6.7. We can find $\tilde{u}_i$ such that

(1) $|\tilde{u}_i - u(x_i, t_i)| = O(1)\,\varepsilon$.

(2) $\tilde{u}_i \in T_i(\tilde{u}_{i-1})$.

(3) $(\tilde{u}_{i-1}, \tilde{u}_i)$ is an i-th elementary wave.

i.e., the superposition of $(\tilde{u}_{i-1}, \tilde{u}_i)$, $i = 1, \cdots, n$ solves

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x, t=0) = \begin{cases} \tilde{u}_0 & x < 0 \\ \tilde{u}_n & x > 0 \end{cases} \end{cases}$$

On the other hand, by definition, (u_{i-1}, u_i) is the i-th elementary of $u_R(x,t) = U\left(\frac{x}{t}\right)$,

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x, t=0) = \begin{cases} u_- = u_0 & x < 0 \\ u_+ = u_n & x > 0 \end{cases} \end{cases}$$

so,

$$|\tilde{u}_0 - u_-| = |\tilde{u}_0 - u_0| = O(1)\,\varepsilon.$$

Similarly,

$$|\tilde{u}_n - u_+| = O(1)\,\varepsilon.$$

By continuous dependence of Riemann solution,

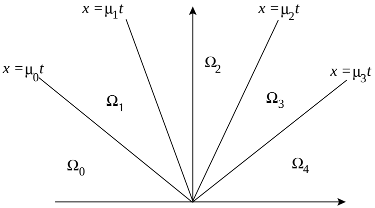
$$\begin{aligned} |\tilde{u}_i - u_i| &= O(1)\,\varepsilon, \quad i = 1, \cdots, n, \\ \text{so} \quad |u(x_i, t_i) - u_i| &= O(1)\,\varepsilon, \quad i = 1, \cdots, n. \end{aligned}$$

Final proof of Theorem 6.1

Proof of (1): $\forall\,\varepsilon > 0$, let X_k^j, Γ_k, Λ_k defined as before, clearly for large enough t, $\Gamma_i \subset \Omega_i$. Furthermore,

$$x = \mu_i\, t \subset \Lambda_i,$$

so for $x = \mu_i\, t$, $u(x,t) = u_i + O(1)\,\varepsilon$. Since ε is arbitrary, so $u(x,t) \to u_i$ on $\dfrac{x}{t} = \mu_i$, as $t \to \infty$.



Proof of (2):

Step 1: $i = \alpha_1, \cdots \alpha_p$, $\nabla\lambda_i \cdot \gamma_i > 0$, (u_{i-1}, u_i) is a centered rarefaction wave, i.e. $\lambda_i(u_i) \geq \lambda_i(u_{i-1})$. Then by Lemma 6.8,

$$\lambda_i(u(x_i, t_i)) - \lambda_i(u(x_{i-1}, t_{i-1})) \geq -O(1)\,\varepsilon$$

$$(x_i, t_i) \in \Lambda_i,\ (x_{i-1}, t_{i-1}) \in \Lambda_{i-1}\ .$$

Then Lemma 6.7 implies

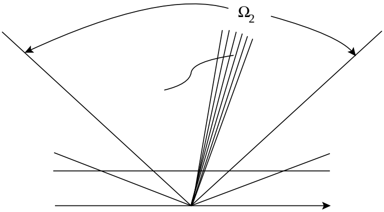
$$|\mathbb{X}_i^-(t)| + |\text{str } X_i(t) = O(1)\,\varepsilon.$$

On the other hand, for large t, $\Gamma_i \subset \Omega_i$, so by Lemma 6.3 that the total amount of i-shock wave in Ω_i is order of $O(\varepsilon)$, which tends to zero as $t \to +\infty$ since ε is arbitrary $\Longrightarrow$ only i-rarefaction waves left.

Step 2: We need to show in fact $u(x,t) \to (u_{i-1}, u_i)$ in Ω_i as $t \to \infty$. By Step 1, $\exists\, t_2 > t_1$ such that $|\mathbb{X}_i^-(t)| + |\text{str } X_i(t)| \le O(1)\,\varepsilon$ and also the speed X_i^1 and X_i^2 are given by $\lambda_i(u_{i-1}) + O(\varepsilon)$ and $\lambda_i(u_i) + O(\varepsilon)$ respectively.

Let $l_i^j (j = 1, 2)$ be the edges of the centered rarefaction wave (u_{i-1}, u_i), $l_i^1 = \{(x,t) |\ \frac{x}{t} = \lambda_i(u_{i-1}\},\ \ l_i^2 = \{(x,t) |\ \frac{x}{t} = \lambda_i(u_i)\}$. Then for $t \ge t_2 + O(1)\, D_i(t_2)$, one has

$$|X_i^1(t) - l_i^1(t)| + |X_i^2(t) - l_i^2(t)| = O(1)\,\varepsilon\,(t - t_2) + O(1).$$



Let $u^*(x,t)$ be the centered rarefaction wave.
Claim: (1)
$|u^*(x,t) - u(x,t)| = O(1)\,\varepsilon, \quad \forall\,(x,t) \in (\Lambda_i \cup \Lambda_{i-1}) \cap \Omega_i.$
(2) $|u^*(x,t) - u(x,t)| = O(1)\,\varepsilon, \quad \forall\,(x,t) \in \Gamma_i \subset \Omega_i.$

(1) $(x,t) \in \Lambda_{i-1} \cap \Omega_i$, then

$$\begin{aligned}
|u^*(x,t) - u(x,t)| &\le |u^*(x,t) - u_{i-1}| + |u_{i-1} - u(x,t)| \\
&= |u^*(x,t) - u^*(l_i^1(t),t)| + O(1)\,\varepsilon \\
&\le O(1)\frac{|l_i^1(t) - X_i^1(t)|}{t} + O(1)\,\varepsilon \\
&\le O(1)\,\varepsilon \qquad \text{for } t \text{ large enough.}
\end{aligned}$$

Now we fix $(x,t) \in \Gamma_i, \;\; t \ge t_2 + O(1)\,D_i(t_2)$. By Step 1, no i-th shocks and other j-waves $(j \ne i)$ (mod $O(\varepsilon)$), and since (1) holds true. $\exists\, x^* \in (l_i^1(t), l_i^2(t))$ such that

$$|u^*(x^*,t) - u(x,t)| = O(\varepsilon).$$

Through (x^*, t) we draw a generalized backward characteristics curve X, its speed changes only when it crosses other waves, since X stays in Γ_i, thus the total amount of other family waves are of the order $O(\varepsilon)$, and when it cross the i-shocks, then its strength is $O(\varepsilon)$, so the speed of X is $\lambda_i(u(x,t)) + O(\varepsilon)$

$$|x^* - x| = O(1)\,\varepsilon\,|t - t_2| + O(1)$$

then

$$\begin{aligned} |u^*(x,t) - u(x,t)| &\leq |u^*(x,t) - u^*(x^*,t)| + |u^*(x^*,t) - u(x,t)| \\ &\leq O(1)|\lambda_i(u^*(x,t)) - \lambda_i(u^*(x^*,t))| + O(\varepsilon) \\ &= O(1)\left|\frac{x - x^*}{t}\right| + O(\varepsilon) \\ &= O(\varepsilon) \qquad \text{for } t \text{ sufficiently large.} \end{aligned}$$