Section 3. Standard Riemann Semigroup Approach

For the Cauchy problem for Burgers equation

$$\begin{cases} \partial_t u + \partial_x \left(\dfrac{u^2}{2} \right) = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}$$

The following contraction principle holds:

If $u_1(x,t)$, $u_2(x,t)$ are "right" solutions to (3.1), then

$$||u_2(x,t) - u_1(x,t)||_{L^1(\mathbb{R}^1)} \leq ||u_1(x,0) - u_2(x,0)||_{L^1(\mathbb{R}^1)} \qquad \forall\, t \geq 0$$

This can generate a semigroup $S_t\, u_0 = u(x,t)$.

B. Temple gave an example to show that one cannot obtain the L_1-contraction principle for systems of conservation laws. However,

Definition 3.1: *(Bressan): The system (2.7) is said to admit a standard Riemann semi-group (SRSG). If for some C_0 and δ_0, there exists a map $S_t : \mathcal{D} \times [0, \infty)' \to \mathcal{D}$ and constant L such that*

(1) $S_0\,\bar{u} = \bar{u}, \qquad \forall\,\bar{u} \in \mathcal{D}$

(2) $S_\tau\,S_s\,\bar{u} = S_{\tau+s}\,\bar{u}, \qquad \forall\,\bar{u} \in \mathcal{D}$

(3) $||S_t\,\bar{u} - S_t\,\bar{\nu}||_{L^1(\mathbb{R})} \leq L||\bar{u} - \bar{\nu}||_{L^1(\mathbb{R})}$

(4) If $\bar{u} \in \mathcal{D}$ and $\bar{u}$ is piecewise with finitely many jumps, then $\exists\, \tau = \tau(\bar{u}) > 0$ such that $\forall\, t \in [0, \tau]$, $(S_t\, \bar{u})(x) = u(x, t)$ must coincide with $u^R(x, t)$ which solves

$$\begin{cases} \partial_t\, u + \partial_x\, f(u) = 0 \\ u(x, t = 0) = \bar{u}(x) \end{cases}$$

by piecing together all Riemann solutions together.

Main Idea: If the system (2.7) admits a SRSG, then every entropy weak solution obtained as a limit of piecewise constant approximate solutions in L^1 must coincide with a trajectory of the SRSG.

Then the following statements are true:

(1) all the trajectories of the SRSG are entropy weak solutions, i.e., S_t must be a solution operator.

(2) SRSG must be unique.

(3) the weak solution obtained by the front tracking method in Proposition 2.2 is unique and depends Lipschits continuously on its initial data.

Theorem 3.1: Assume that (2.7) admits a SRSG, $S : \mathcal{D} \times [0, \infty) \to \mathcal{D}$. Let $\{u^\nu\}$ be a sequence of approximate solutions constructed by the front tracking method, as given by Proposition 2.1, with $\varepsilon_i \to 0$, $\nu_i \to \infty$, suppose that

$$\begin{array}{rcll} u^\nu(\cdot, 0) & \longrightarrow & u_0(\cdot) & \text{in} \quad L^1 \\ u^\nu & \longrightarrow & u & \text{in} \quad L^1_{loc}(\mathbb{R}^1 \times \mathbb{R}^1_+;\ \mathbb{R}^n) \end{array}$$

for $u_0 \in \mathcal{D}$. Then

$$u(x, t) = S_t\, u_0.$$

To prove this theorem, we need two lemmas.

Lemma 3.1: Let $u_\pm \in \Omega$ and $|\lambda| < \bar{\lambda}$. Let $w(x,t)$ be the self-similar solutions

$$\begin{cases} \partial_t w + \partial_x f(w) = 0 \\ w(x,t) = \begin{cases} u_- & x < 0 \\ u_+ & x > 0 \end{cases} \end{cases}$$

Set

$$\nu(x,t) = \begin{cases} u_- & \frac{x}{t} < \lambda \\ u_+ & \frac{x}{t} > \lambda \end{cases}$$

Then

(1) $\frac{1}{t}\int_{-\infty}^{+\infty}|\nu(x,t)-w(x,t)|dx = O(1)|u_+ - u_-|$

(2) If, in addition, $u_+ = \exp\{\sigma\gamma_i\}u_-$, $\lambda = \lambda_i(u_+)$ for $\sigma > 0$, $i \in \{1,\cdots,n\}$, then

$$\frac{1}{t}\int_{-\infty}^{\infty}|\nu(x,t)-w(x,t)|dx = O(1)\sigma^2$$

(3) If $\lambda = \tilde{\lambda}_i$, $\tilde{\lambda}_i$ is an eigenvalue of $f'(\tilde{u})$, and $\nabla f(\tilde{u})(u^+ - u^-) = \lambda(\tilde{u})(u^+ - u^-)$, then

$$\frac{1}{t}\int_{-\infty}^{\infty}|\nu(x,t)-w(x,t)|dx = O(1)\,|u_+ - u_-|\,(|u_+ - \tilde{u}| + |\tilde{u} - u_-|)$$

Lemma 3.2: Let S be SRSG. Let $u : \mathcal{D} \times [0, +\infty) \to \mathcal{D}$ whose values are piecewise constant with finitely many polygonal lines, say $x_\alpha(t)$, $\alpha = 1, \cdots, m$, then

$$||u(\cdot, T) - S_T\, u(\cdot, 0)||_{L^1} \leq L \int_0^T \overline{\lim}_{h \to 0_+} \frac{||u(t+h) - S_h\, u(t)||_{L^1}}{h}\, dt \;(\bigstar)$$

Remark: There is a localized version of $(\bigstar)$ due to finite speed of propagation. For any given a, b, $b > a$, constants, define
$I_t = (a + \bar{\lambda} t, b - \bar{\lambda} t)$ for $t < \dfrac{b-a}{2\bar{\lambda}}$,

$$||u(t) - S_t\, u(0)||_{L^1(I_t)} \leq L \int_0^t \overline{\lim}_{h \to 0_+} \frac{||u(\tau + h) - S_h\, u(\tau)||_{L^1(I_{\tau+h})}}{h} d\tau$$

We assume Lemma 3.1 and Lemma 3.2 for a moment and continue the proof.

Proof of Theorem 3.1 It suffices to show

$$||u(T) - S_T\, u_0||_{L^1} = 0 \quad \text{for any } \; T > 0.$$

This is equivalent to say $\overline{\lim}_{\nu \to +\infty} ||u^\nu(T) - S_T\, u_0||_{L^1} = 0$. Note that

$$\begin{aligned} ||u^\nu(T) - S_T\, u_0||_{L^1} &\leq ||u^\nu(T) - S_T\, u_0^\nu||_{L^1} + ||S_T\, u_0^\nu - S_T\, u_0||_{L^1} \\ &\leq ||u^\nu(T) - S_T\, u_0^\nu||_{L^1} + L\, ||u_0^\nu - u_0||_{L^1} \end{aligned}$$

It will suffice to prove that

$$\overline{\lim}_{\nu \to +\infty} ||u^\nu(T) - S_T\, u_0^\nu||_{L^1} = 0$$

However, by Lemma 3.2,

$$||u^\nu(T) - S_T\, u_0^\nu||_{L^1} \le L \int_0^T \overline{\lim}_{h\to 0_+} \frac{||u^\nu(t+h) - S_h\, u^\nu(t)||_{L^1}}{h}\, dt$$

It suffices to compute that

for $t \in [0, T]$, $\overline{\lim}_{h\to 0_+} \dfrac{||u^\nu(t+h) - S_h\, u^\nu(t)||_{L^1}}{h}$.

$$\begin{aligned}
\text{Let } \mathcal{S} &= \{\alpha \in \{1, \cdots, m\}, \text{ such that } u^\nu(x_\alpha^-, t) = \\
&\quad u_-^\nu, u^\nu(x_\alpha^+, t) = u_+^\nu \text{ are connected either} \\
&\quad \text{by shock wave or contact discontinuity}\}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{R} &= \{\alpha \in \{1, \cdots, m\}, \text{ such that } u_-^\nu \text{ and } u_+^\nu \text{ corresponding to a} \\
&\quad \text{rarefaction wave in the } k_\alpha-\text{th family, so that} \\
&\quad \dot{x}_\alpha(t) = \lambda_{k_\alpha}(u_+^\nu),\ u_+^\nu = \exp(\varepsilon_\alpha\, \gamma_{k_\alpha}) u_-^\nu,\ \varepsilon_\alpha \in [0, \varepsilon]\}.
\end{aligned}$$

Set $\alpha \in \mathcal{S} \cup \mathcal{R}$, $w^\alpha(x,t)$ satisfies

$$\begin{cases} \partial_t w^\alpha + \partial_x f(w^\alpha) = 0 \\ w^\alpha(x, t=0) = \begin{cases} u_- = u^\nu(x_\alpha^-, t) \\ u_+ = u^\nu(x_\alpha^+, t) \end{cases} \end{cases}$$

Define

$$\begin{cases} \partial_t w^\beta + \partial_x f(w^\beta) = 0 \\ w^\beta(x, t=0) = \begin{cases} u_- = u^\nu(x_\beta^-, t) \\ u_+ = u^\nu(x_\beta^+, t) \end{cases} \end{cases}$$

where $x = x_\beta(t)$ is associated with a pseudo-shock.

Note that if $t \in [0, T]$ such that t is NOT a node point (i.e. t is not a time two of x_α's interact)

$$u(x, t+h) - S_h\, u(x,t) = 0$$

in a region away from a ρ-neighborhood of $x_\alpha \in \mathcal{R}$ and x_β for a non-physical wave for some $\rho > 0$ small. Therefore

$$\begin{aligned}
& \overline{\lim}_{h\to 0+} \ \frac{||u^\nu(t+h) - S_h\, u^\nu(t)||}{h} \\
= \quad & \sum_{\alpha\in\mathcal{R}} \lim_{h\to 0} \frac{1}{h} \int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} |u^\nu(t+h) - w^\alpha(x - x_\alpha(t), h)| \, dx \\
& \qquad + \sum_{\beta} \frac{1}{h} \int_{x_\beta(t)-\rho}^{x_\beta(t)+\rho} |u^\nu(t+h) - w^\beta(x - x_\beta(t), h)| \, dx \\
= \quad & C \sum_{\alpha\in\mathcal{R}} |\varepsilon_\alpha|^2 + C \cdot \sum_{\beta} |u_\nu^- - u_\nu^+| \\
\leq \quad & C \cdot \varepsilon_\nu \sum_{\alpha\in\mathcal{R}} |\varepsilon_\alpha| + C \cdot \sum_{\beta} |u_\nu^- - u_\nu^+| \\
\leq \quad & \varepsilon_\nu (C \cdot T.V.u^\nu + C_1) \\
\longrightarrow \quad & 0 \qquad \text{as} \qquad \nu \to \infty
\end{aligned}$$

Proof of Lemma 3.1

(1) $$\frac{1}{t}\int_{-\infty}^{\infty} |w(x,t)-\nu(x,t)|dx = \frac{1}{t}\int_{-\bar{\lambda}t}^{\bar{\lambda}t} |w(x,t)-\nu(x,t)|dx$$
$$= \frac{1}{t}\int_{-\bar{\lambda}t}^{\bar{\lambda}t} O(1)\ |u^- - u^+|dx = O(1)\ |u^- - u^+|\bar{\lambda}.$$

(2) $u^+ = \exp\{\sigma\gamma_i\}(u^-), \quad \sigma \geq 0, \quad \lambda = \lambda_i(u^+)$

Case 1: The i-th family is linearly degenerate, w is a contact discontinuity with speed $\lambda = \lambda_i(u^+) = \lambda_i(u^-)$,
$w(x,t) = \nu(x,t)$.

Case 2: The i-th family is genuinely nonlinear. $\sigma > 0$, $w(x,t)$ is an i-th rarefaction wave.

$$
\begin{aligned}
& \frac{1}{t}\int_{-\infty}^{\infty} |w(x,t)-\nu(x,t)|dx \\
= \ & \frac{1}{t}\int_{-\infty}^{\lambda_i(u^-)t} |w(x,t)-\nu(x,t)|dx \\
& + \frac{1}{t}\int_{\lambda_i(u^-)t}^{\lambda_i(u^+)t} |w(x,t)-\nu(x,t)|dx \\
& + \frac{1}{t}\int_{\lambda_i(u^+)t}^{+\infty} |w(x,t)-\nu(x,t)|dx \\
= \ & \frac{1}{t}\int_{\lambda_i(u^-)t}^{\lambda_i(u^+)t} |w(x,t)-\nu(x,t)|dx \\
= \ & O(1)\ |u^+ - u^-| \cdot |\lambda_i(u^+) - \lambda_i(u^-)| \\
= \ & O(1)\sigma^2
\end{aligned}
$$

(3) $\nabla f(\tilde{u})(u^+ - u^-) = \tilde{\lambda}_i(u^+ - u_-)$, $\tilde{\lambda}_i = \lambda_i(\tilde{u})$, therefore $u^+ - u^- = \theta\, r_i(\tilde{u})$.

Solving Riemann problem, $\exists\ w_0, w_1, \cdots, w_n$ such that $w_0 = u_-$, $w_n = u_+$, such that w_j is connected w_{j-1} by an elementary wave. Let the strength of the j-th wave in $w(x,t)$ be $\sigma_j(\theta, \tilde{u})$.

If $\tilde{u} = u^-$, $\left.\dfrac{\partial\, \sigma_j(\theta, u^-)}{\partial\, \theta}\right|_{\theta=0} = \delta_{ij}$

$$\left|\frac{\partial\, \sigma_j(\theta, \tilde{u})}{\partial\, \theta} - \delta_{ij}\right| \leq O(1)\ (|\theta| + |u^- - \tilde{u}|)$$

Claim:

(i) $|\sigma_j(\theta, \tilde{u})| \le O(1)\ |\theta|\ (|u^+ - \tilde{u}| + |u^- - \tilde{u}|)$ for $j \ne i$;

(ii)
$$\begin{aligned} &\max(|w_i - u^+|, |w_{i-1} - u^-|) \\ &\le O(1)\ |\theta|\ (|u^+ - \tilde{u}| + |u^- - \tilde{u}|); \end{aligned}$$

(iii)
$$\begin{aligned} &\max\left\{|\lambda_i(w_i) - \tilde{\lambda}_i|, |\lambda_i(w_{i-1}) - \tilde{\lambda}_i|\right\} \\ &\le O(1)\ (|u^+ - \tilde{u}| + |u^- - \tilde{u}|). \end{aligned}$$

Thus

$$\frac{1}{t}\int_{-\infty}^{\infty} |w(x,t) - \nu(x,t)| dx = \frac{1}{t}\int_{-\bar{\lambda}t}^{\bar{\lambda}t} |w(x,t) - \nu(x,t)| dx$$

Proof of Lemma 3.2

Step 1: Let the node points be at times $\tau_1, \tau_2, \cdots, \tau_m$. Note that except at τ_i, then

$$\overline{\lim}_{h\to 0+} \frac{||u(t+h) - S_h\, u(t)||_{L^1}}{h} \text{ is constant, } t \in (\tau_i, \tau_{i+1}).$$

Step 2: $\forall \varepsilon > 0$, fixed, let τ be defined as

$$\begin{aligned} \tau \ = \ & \max\{t \in [0, T] \mid \ \text{such that} \ \ ||S_{T-t}\, u(t) - S_T\, u(o)||_{L^1} \\ & \leq L\left(\varepsilon t + \int_0^t \lim_{h\to 0+} \frac{||u(s+h) - S_h\, u(s)||_{L^1}}{h} ds\right) \\ & + \varepsilon \sum_{\tau_i < t} 2^{-i}\} \end{aligned}$$

Fact 1: τ exists 1. Since $A \neq \phi$, due to $t = 0 \in A$.

Fact 2: $\tau \in A$, because the left hand side of the above expression is continuous, and right hand side is lower semi-continuous.

Fact 3: Lemma will be proved if we can show $\tau = T$.

Claim: $\tau = T$.

If not, $\tau < T$.

Case 1: $\exists\, j$, such that $\tau = \tau_j$.
Then by the continuity of semi-group S, $\exists\, \delta > 0$ such that

$$
\begin{aligned}
& ||S_{T-t}\, u(t) - S_T\, u(0)||_{L^1} \\
\leq\ & ||S_{T-\tau}\, u(\tau) - S_T\, u(0)||_{L^1} + \varepsilon\, 2^{-j} \qquad t \in [\tau, \tau+\delta] \\
\leq\ & L \left(\varepsilon\, \tau + \int_0^\tau \overline{\lim}_{h\to 0+} \frac{||u(s+h) - S_h\, u(s)||_{L^1}}{h}\, ds \right) \\
& + \varepsilon \sum_{\tau_i < \tau} 2^{-i} + \varepsilon\, 2^{-j} \\
\leq\ & L \left(\varepsilon\, t + \int_0^t \overline{\lim}_{h\to 0+} \frac{||u(s+h) - S_h\, u(s)||_{L^1}}{h}\, ds \right) \\
& + \varepsilon \sum_{\tau_i < t} 2^{-i}
\end{aligned}
$$

This implies $t \in A$ contradiction.

Case 2: $\tau \notin \{\tau_i,\ i = 1, 2, \cdots, m\}$

$\exists\, \delta^* > 0$, such that

$$\overline{\lim}_{h \to 0+} \frac{||S_h\, u(s) - u(s+h)||_{L^1}}{h} = \text{const.} \qquad \text{for } S \in [\tau, \tau + \delta^*]$$

We now choose δ^* small enough such that

$$\frac{||u(\tau+\delta) - S_\delta\, u(\tau)||}{\delta} \leq \varepsilon + \overline{\lim}_{h \to 0+} \frac{||u(\tau+h) - S_h\, u(\tau)||}{h}, \delta \in [0, \delta^*]$$

$$
\begin{aligned}
& ||S_{T-(\tau+\delta)}\, u(\tau+\delta) - S_T\, u(0)||_{L^1} \\
\leq\ & ||S_{T-(\tau+\delta)}\, u(\tau+\delta) - S_{T-(\tau+\delta)}\, S_\delta\, u(\tau)||_{L^1} + ||S_{T-\tau}\, u(\tau) - S_T\, u(0)||_{L^1} \\
\leq\ & L||u(\tau+\delta) - S_\delta\, u(\tau)||_{L^1} \\
& + L\left(\varepsilon\,\tau + \int_0^\tau \lim_{h\to 0+} \frac{||u(s+h) - S_h\, u(s)||_{L^1}}{h}\ ds\right) + \varepsilon \sum_{\tau_i<\tau} 2^{-i} \\
\leq\ & L\,\delta\left(\varepsilon + \overline{\lim}_{h\to 0+} \frac{||u(\tau+h) - S_h\, u(\tau)||_{L^1}}{h}\right) \\
& + L\left(\varepsilon\,\tau + \int_0^\tau \lim_{h\to 0+} \frac{||u(s+h) - S_h\, u(s)||_{L^1}}{h}\right) + \varepsilon \sum_{\tau_j<\tau+\delta} 2^{-j} \\
=\ & L\left(\varepsilon(\tau+\delta) + \int_0^{\tau+\delta} \overline{\lim}_{h\to 0+} \frac{||S_h\, u(s) - u(s+h)||_{L^1}}{h}\ ds\right) \\
& + \varepsilon \sum_{\tau_j<\tau+\delta} 2^{-j}
\end{aligned}
$$

Remark: The standard Riemann Semigroup for (2.7) exists. The existence follows from the construction of front tracking method summarized in Proposition 2.1.