Suggested Solution to Assignment 5

Exercise 5.1

2. (a)

$$A_m = 2\int_0^1 x^2 \sin m\pi x\ dx = -2\frac{x^2}{m\pi}\cos m\pi x\Big|_0^1 + \int_0^1 \frac{4x}{m\pi}\cos m\pi x\ dx$$
$$= \frac{2(-1)^{m+1}}{m\pi} + \frac{4(-1)^m - 4}{m^3\pi^3}.$$

(b)

$$A_m = 2\int_0^1 x^2 \cos m\pi x dx = 2\frac{x^2}{m\pi}\sin m\pi x\Big|_0^1 - \int_0^1 \frac{4x}{m\pi}\sin m\pi x dx = (-1)^m\frac{4}{m^2\pi^2}. \qquad \square$$

4. To find the Fourier series of the function $f(x) = |\sin x|$, we first note that this is an even function so that it has a cos-series. If we integrate from 0 to π and multiply the result by 2, we can take the function $\sin x$ instead of $|\sin x|$ so that

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x dx = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x\cos nx dx = \begin{cases} \frac{4}{(1-n^2)\pi} & n\,\text{even} \\ 0 & n\,\text{odd} \end{cases}.$$

Hence, we have

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right).$$

Substituting $x = 0$ and $x = \frac{\pi}{2}$, we have

$$\sum_{n=1}^\infty \frac{1}{4n^2-1} = \frac{1}{2}.$$

$$\sum_{n=1}^\infty \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}. \qquad \square$$

5. (a) From Page.109, we have

$$x = \sum_{m=1}^\infty (-1)^{m+1}\frac{2l}{m\pi}\sin\frac{m\pi x}{l}.$$

Integration of both sides gives

$$\frac{x^2}{2} = c + \sum_{m=1}^\infty (-1)^m \frac{2l^2}{m^2\pi^2}\cos\frac{m\pi x}{l}.$$

The constant of the integration is the missing coefficient

$$c = \frac{A_0}{2} = \frac{1}{l}\int_0^l \frac{x^2}{2}dx = \frac{l^2}{6}.$$

(b) By setting $x = 0$ gives

$$0 = \frac{l^2}{6} + \sum_{m=1}^\infty (-1)^m\frac{2l^2}{m^2\pi^2},$$

or

$$\frac{\pi^2}{12} = \sum_{m=1}^\infty \frac{(-1)^{m+1}}{m^2}. \qquad \square$$

8. The key point in the problem above is to solve the following PDE problem.

$$u_t - u_{xx} = 0, \quad u(x,0) = \phi(x), \quad u(0,t) = u(l,t) = 0,$$

$$\phi(x) = \begin{cases} \frac{3}{2}, & 0 < x < \frac{2}{3}, \\ 3 - 3x, & \frac{2}{3} < x < 1 \end{cases}.$$

Through a standard procedure of separation variable method, we obtain

$$u(x,t) = \sum a_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where $a_n = 2\int_0^1 \phi(x)\sin n\pi x dx = \frac{9}{n^2\pi^2}\sin\frac{2\pi n}{3}$, so the solution $T = u(x,t) + x$. □

9. From Section 4.2.7, we see that the general formula to wave equation with Neu- mann boundary condition is

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty}(A_n \cos nct + B_n \sin nct)\cos nx,$$

where

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx, \quad \psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} ncB_n \cos nx.$$

By further calculation, we have $B_0 = 1$, $B_2 = \frac{1}{4c}$ and the other coefficients are all zero. Hence, the solution is

$$u(x,t) = \frac{1}{2}t + \frac{\sin 2ct \cos 2x}{4c}. \quad \square$$

Exercise 5.2

2. Suppose $\alpha = p/q$, where p, q are co-prime to each other. Then is is not difficult to see that $S = 2q\pi$ is a period of the function. Suppose $2q\pi = mT$, where T is the minimal period. Then

$$\cos x + \cos \alpha x = \cos(x+T) + \cos(\alpha x + \alpha T).$$

Let $x = 0$, we have the above equality holds iff q/m, p/m are both integers. Therefore, $m = 1$. Hence, we finish the problem. □

5. Let $a_m = \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}$. Then we have

$$\phi(x) = \sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{l}. \quad \square$$

8. (a) If f is even, $f(-x) = f(x)$. Differentiating both sides gives $-f'(-x) = f'(x)$, so $f'(-x) = -f'(x)$, showing f' is odd. If f is odd, $f(-x) = -f(x)$. Differentiating both sides gives $-f'(-x) = -f'(x)$, so $f'(-x) = f'(x)$, showing f' is even.

   (b) If f is even, consider $\int f(-x)dx = \int f(x)dx$. Via substitution, $u = -x$, we have $-\int f(u)du = \int f(x)dx$. So if ignoring te constant of integration, $F(-x) = -F(x)$, showing F is odd, where F is an antiderivative of f.Similarly, for f odd, we have $\int f(-x)dx = -f(x)dx$, so $F(-x) = F(x)$, showing F is even. □

10. (a) If ϕ is continuos on $(0,l)$, ϕ_{odd} is continuous on $(-l,l)$ if and only if $\lim_{x\to 0^+}\phi(x) = 0$.

    (b) If $\phi(x)$ is differentiable on $(0,l)$, ϕ_{odd} is differentiable on $(-l,l)$ if and only if $\lim_{x\to 0^+}\phi'(x)$ exists, since ϕ'_{odd} is an even function, so the only thing to avoid is an infinite discontinuity at $x = 0$.

(c) If ϕ is continuos on $(0,l)$, ϕ_{even} is continuous on $(-l,l)$ if and only if $\lim_{x\to 0^+}\phi(x)$ exists, since the only thing to avoid is an infinite discontinuity at $x=0$.

(d) If $\phi(x)$ is differentiable on $(0,l)$, ϕ_{even} is differentiable on $(-l,l)$ if and only if $\lim_{x\to 0^+}\phi'(x)=0$, since ϕ'_{even} is an odd function. □

Extra. $u(0,t)=u(1,t)=0$ tells us we can do odd extension and periodic extension with period 2. Thus define

$$\phi(x)=\begin{cases}\sin^2(\pi x), & x\in[2n,2n+1]\\ -\sin^2(\pi x), & x\in[2n-1,2n]\end{cases}$$

$$\psi(x)=\begin{cases}x(1-x), & x\in[2n,2n+1]\\ x(1+x), & x\in[2n-1,2n]\end{cases}$$

$n=0,\pm1,\pm2,....$ By d'Alembert's formula,$u(x,t)=\frac{1}{2}[\phi(x+2t)+\phi(x-2t)]+\frac{1}{4}\int_{x-2t}^{x+2t}\psi(s)ds$ solves the problem.

Exercise 5.3

3. Since $X(0)=0$, by the odd extension $x(-x)=-X(x)$ for $-l<x<0$, then X satisfies $X''+\lambda X=0$, $X'(-l)=X'(l)=0$. Hence,

$$\lambda=[(n+\frac{1}{2})\pi]^2/l^2,\ X_n(x)=\sin[(n+\frac{1}{2})\pi x/l],\ n=0,1,2,\ldots$$

Thus we botain the general formula to this equation

$$u(x,t)=\sum_{n=0}^{\infty}[A_n\cos\frac{(n+\frac{1}{2})\pi ct}{l}+B_n\sin\frac{(n+\frac{1}{2})\pi ct}{l}]\sin\frac{(n+\frac{1}{2})\pi x}{l}.$$

By the boundry condition, we obtained that B_n are all zero, while $A_n=\frac{2}{l}\int_0^l\sin\frac{(n+\frac{1}{2})\pi x}{l}\cdot x\ dx=(-1)^n\frac{2l}{(n+\frac{1}{2})^2\pi^2}$.

5(a). Let $u(x,t)=X(x)T(t)$, then

$$-X''(x)=\lambda X(x),$$
$$X(0)=0,\ X'(l)=0.$$

By Theorem 3, there is no negative eigenvalue. It is easy to check that 0 is not an eigenvalue. Hence, there are only positive eigenvalues.
Let $\lambda=\beta^2$, $\beta>0$, then we have

$$X(x)=A\cos\beta x+B\sin\beta x.$$

Hence the bounndary condtions imply

$$A=0,\ B\beta\cos\beta l=0.$$

$$\beta=\frac{(n+\frac{1}{2})\pi}{l},\ n=0,1,2,\ldots$$

So the eigenfunctions are

$$X_n(x)=\sin\frac{(n+\frac{1}{2})\pi x}{l},\ n=0,1,2,\ldots \qquad \square$$

6. Let $X'(x) = \lambda X(x)$, $\lambda \in \mathbb{C}$, then
$$X(x) = e^{\lambda x}.$$
By the boundary condition $X(0) = X(1)$, we have
$$e^{\lambda} = 1.$$
Hence,
$$\lambda_n = 2n\pi i,\ X_n(x) = e^{2n\pi x i},\ n \in \mathbb{Z}.$$
Since, if $m \neq n$,
$$\int_0^1 X_n(x)\overline{X_m(x)}dx = \int_0^1 e^{2(n-m)\pi x i}dx = 0.$$
Therefore, the eigenfunctions are orthogonal on the interval $(0,1)$. □

8. If
$$X_1'(a) - a_a X_1(a) = X_2'(a) - a_a X_2(a) = 0,$$
and
$$X_1'(b) + a_b X_1(b) = X_2'(b) + a_b X_2(b) = 0,$$
then
$$\begin{aligned}(-X_1'X_2 + X_1X_2')|_a^b &= -X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(a) - X_1(a)X_2'(a) \\ &= a_bX_1(b)X_2(b) - X_1(b)a_bX_2(b) + a_aX_1(a)X_2(a) - X_1(a)a_aX_2(a) = 0.\end{aligned}$$ □

9. For $j = 1, 2$, suppose that
$$\begin{aligned}X_j(b) &= \alpha X_j(a) + \beta X_j'(a)\\ X_j'(b) &= \gamma X_j(a) + \delta X_j'(a).\end{aligned}$$
Then,
$$\begin{aligned}(X_1'X_2 - X_1X_2')|_a^b &= X_1'(b)X_2(b) - X_1(b)X_2'(b) - X_1'(a)X_2(a) + X_1(a)X_2'(a)\\ &= [\gamma X_1(a) + \delta X_1'(a)][\alpha X_2(a) + \beta X_2'(a)]\\ &\quad - [\alpha X_1(a) + \beta X_1'(a)][\gamma X_2(a) + \delta X_2'(a)] - X_1'(a)X_2(a) + X_1(a)X_2'(a)\\ &= (\alpha\delta - \beta\gamma - 1)X_1'(a)X_2(a) + (1 + \beta\gamma - \alpha\delta)X_1(a)X_2'(a)\\ &= (\alpha\delta - \beta\gamma - 1)(X_1X_2)'|_{x=a}.\end{aligned}$$
Therefore, the boundary conditions are symetric if and only if $\alpha\delta - \beta\gamma = 1$. □

12. By the divergence theorem,
$$f'g|_a^b = \int_a^b (f'(x)g(x))'dx = \int_a^b f''(x)g(x) + f'(x)g'(x)dx,$$
$$\int_a^b f''(x)g(x)dx = -\int_a^b f'(x)g'(x)dx + f'g|_a^b.$$ □

13. Substitute $f(x) = X(x) = g(x)$ in the Green's first identity, we have
$$\int_a^b X''(x)X(x)dx = -\int_a^b X'^2(x)dx + (X'X)|_a^b \leq 0.$$
Since $-X'' = \lambda X$, so
$$-\lambda \int_a^b X^2(x)dx \leq 0.$$
Therefore, we get $\lambda \geq 0$ since $X \not\equiv 0$. □

Exercise 5.4

1. The partial sum is given by
$$S_n = \frac{1-(-1)^n x^{2n}}{1+x^2}.$$
   (a) Obviously for any x_0 fixed, $S_n \to \frac{1}{1+x_0^2}$. Thus it converges to $\frac{1}{1+x^2}$ pointwise.
   (b) Let $x_n = 1 - \frac{1}{n}$, then $x^{2n} \to e^{-2}$. Thus it does not converge uniformly.
   (c) It will converge to $S(x) = \frac{1}{1+x^2}$ in the L^2 sence since
$$\begin{aligned}\int_{-1}^{1} |S_n - S|^2 dx &= \int_{-1}^{1} \frac{x^{4n}}{(1+x^2)^2} dx \\ &\le \int_{-1}^{1} x^{4n} dx \\ &\le \frac{2}{4n+1} \to 0 \quad \text{as } n \to \infty. \qquad \square\end{aligned}$$

2. This is an easy consequence combined Theorem 2 and Theorem 3 on Page 124 and Theorem 4 on Page 125. $\square$

3. (a) For any fixed point x_0, WLOG, we assume $x_0 < \frac{1}{2}$. Then there is N_0 such that for $n > N_0$,
$$x_0 < \frac{1}{2} - \frac{1}{n},$$
   which implies that $f_n(x_0) \equiv 0$. Thus $f_n(x) \to 0$ pointwisely.
   (b) Let $x_n = \frac{1}{2} - \frac{1}{n}$, then $f_n(x_n) = -\gamma_n \to -\infty$, which implies that the convergence is not uniform.
   (c) By direct computation, we have
$$\int f_n^2(x) dx = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \gamma_n^2 dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \gamma_n^2 dx = \frac{2\gamma_n^2}{n}.$$
   For $\gamma_n = n^{\frac{1}{3}}$,
$$\int f_n^2(x) dx = 2n^{-\frac{1}{3}} \to 0 \quad \text{as } n \to \infty.$$
   (d) By the computation in (c), for $\gamma_n = n$,
$$\int f_n^2(x) dx = 2n \to \infty \quad \text{as } n \to \infty. \qquad \square$$

4. For odd n,
$$\int_{\frac{1}{4}-\frac{1}{n^2}}^{\frac{1}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0.$$
   For even n,
$$\int_{\frac{3}{4}-\frac{1}{n^2}}^{\frac{3}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0.$$
   Thus, for any n,
$$\|g_n(x)\|_{L^2}^2 = \frac{2}{n^2} \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

5. (a) We see that $A_0 = \frac{2}{3}\int_1^2 dx = \frac{4}{3}$ and $A_m = \frac{2}{3}\int_2^3 \cos\frac{m\pi x}{3} dx = -\frac{2}{mx}\sin\frac{m\pi}{3}$. So, the first four nonzero terms are $\frac{4}{3}$, $-\frac{\sqrt{3}}{pi}\cos\frac{\pi x}{3}$, $-\frac{\sqrt{3}}{2\pi}\cos\frac{2\pi x}{3}$ and $\frac{\sqrt{3}}{4\pi}\cos\frac{4\pi x}{3}$.

   (b) We can express $\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{3} + B_n \sin\frac{n\pi x}{3})$. by Theorem 4 of Sectiion 4, since $\phi(x)$ and its derivative is piecewise continuous, so we get the fourier series will converge to $f(x)$ except at $x = 1$, while the value of this series at $x = 1$ is $\frac{1}{2}$.

   (c) By corollary 7, we see that it converge to $\phi(x)$ in L^2 sense.

   (d) Put $x = 0$, we see that the sine series vanish, it turns out to be that $\phi(0) = \frac{2}{3} - \frac{\sqrt{3}}{\pi}\sum_{1\le m<\infty, m\neq 3n} \frac{(-1)^{[\frac{m}{3}]}}{m}\cos\frac{m\pi 0}{3}$ thus we obtain the sum of thee series is $\frac{2\pi}{3\sqrt{3}}$. □

6. The series is $\cos x = \sum_{n=1}^{\infty} a_n \sin nx$. If $n > 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx dx = -\frac{1}{\pi}[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}]\Big|_0^{\pi} = \frac{2n(1+(-1)^n)}{(n^2-1)\pi}.$$

   If $n = 1$, $a_1 = 0$. The sum series is 0 if $x = -\pi, 0, \pi$. By Theorem 4 in Section 4, the sum series converges to $\cos x$ pointwisely in $0 < x < \pi$, and to $-\cos x$ for $-\pi < x < 0$. □

7. (a) Obviously $\phi(x)$ is odd. Thus, its full Fourier series is just the Sine Fourier series, i.e.

$$\sum_{n=1}^{\infty} B_n \sin n\pi x,$$

   where B_n satisfies

$$B_n = \int_{-1}^{1} \phi(x)\sin n\pi x dx = \frac{2}{n\pi}.$$

   (b) By (a), the first three nonzero terms are

$$\frac{2}{\pi}\sin\pi x,\ \frac{1}{\pi}\sin 2\pi x,\ \frac{2}{3\pi}\sin 3\pi x.$$

   (c) Since

$$\int_{-1}^{1}|\phi(x)|^2 dx = 2\int_0^1 (1-x)^2 dx \le 2,$$

   it cconverges in the mean square sense according to Corollary 7.

   (d) Since $\phi(x)$ is continuous on $(-1, 1)$ except at the point $x = 0$. Therefore, Theorem 4 in Section 4 implies it converges pointwisely on $(-1, 1)$ expect at $x = 0$.

   (e) Since the series does not converge pointwisely, it does not converge uniformly.

Exercise 5.6

1. (a) (Use the method of shifting the data.)
   Let $v(x,t) := u(x,t) - 1$, then v solves

$$v_t = v_{xx},\ v_x(0,t) = v(1,t) = 0,\ \text{and}\ v(x,0) = x^2 - 1.$$

   By the method of seperation of variables, we have

$$v(x,t) = \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2\pi^2 t}\cos[(n+\frac{1}{2})\pi x],$$

where

$$A_n = (-1)^{n+1} 4(n + \frac{1}{2})^{-3} \pi^{-3}.$$

Hence,

$$u(x,t) = 1 + \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2 \pi^2 t} \cos[(n + \frac{1}{2})\pi x],$$

where A_n is as before.

(b) 1. □

2. In the case $j(t) = 0$ and $h(t) = e^t$, by (10) and the initial condition $u_n(0) = 0$,

$$u_n(t) = \frac{2n\pi k}{(\lambda_n k + 1)l^2}(e^t - e^{-\lambda_n k t}).$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2n\pi k}{(\lambda_n k + 1)l^2}(e^t - e^{-\lambda_n k t}) \sin \frac{n\pi x}{l}. \qquad \square$$

5. It is easy to check that $\dfrac{e^t \sin 5x}{1 + 25c^2}$ solves

$$v_t t = c^2 v_{xx} + e^t \sin 5x, \qquad \text{and} \qquad v(0,t) = v(\pi,t) = 0.$$

Using the method of shifting the data, we have

$$u(x,t) = \frac{e^t \sin 5x}{1 + 25c^2} + \sum_{n=1}^{\infty} (A_n \cos(nct) + B_n \sin(nct)) \sin(nx),$$

where

$$A_n = -\frac{2}{\pi} \int_0^{\pi} \frac{1}{1 + 25c^2} \sin 5x \sin nx \; dx = \begin{cases} -\dfrac{1}{1+25c^2} & n = 5 \\ 5 & \text{otherwise} \end{cases};$$

$$B_n = \frac{2}{nc\pi} \int_0^{\pi} [\sin 3x - \frac{1}{1+25c^2} \sin 5x] \sin nx \; dx$$

$$= \begin{cases} \frac{1}{3c} & n = 3 \\ -\dfrac{1}{5c(1+25c^2)} & n = 5 \\ 0 & \text{otherwise} \end{cases}.$$

So the formula of the solution can be simplfied as

$$u(x,t) = \frac{1}{3c} \sin 3ct \sin 3x + \frac{1}{1+25c^2} \left(e^t - \cos 5ct - \frac{1}{5c} \sin 5ct \right) \sin 5x. \qquad \square$$

8. (Expansion Method) Let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l},$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi x}{l},$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin \frac{n\pi x}{l}.$$

Then

$$
\begin{aligned}
v_n(t) &= \frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\sin\frac{n\pi x}{l}dx = \frac{du_n}{dt},\\
w_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{l}dx = \frac{du_n}{dt},\\
&= -\frac{2}{l}\int_0^l (\frac{n\pi}{l})^2 u(x,t)\sin\frac{n\pi x}{l}dx + \frac{2}{l}(u_x \sin\frac{n\pi x}{l} - \frac{n\pi}{l}u\cos\frac{n\pi x}{l})\Big|_0^l\\
&= -\lambda_n u_n(t) - 2n\pi l^{-2}(-1)^n At,
\end{aligned}
$$

where $\lambda_n = (n\pi/l)^2$. Here we used the Green's second identity and the boundary conditions. Hence, by the PDE $u_t = ku_{xx}$ and the initial condition $u(x,0) = 0$, we get

$$\frac{du_n}{dt} = k[-\lambda_n u_n(t) - 2n\pi l^{-2}(-1)^n At],$$

$$u_n(0) = 0.$$

Hence,

$$u_n(t) = (-1)^{n+1}2n\pi l^{-2}A[\frac{t}{\lambda_n} - \frac{1}{\lambda_n^2 k} + \frac{e^{-\lambda_n kt}}{\lambda_n^2 k}].$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty}(-1)^{n+1}2n\pi l^{-2}A[\frac{t}{\lambda_n} - \frac{1}{\lambda_n^2 k} + \frac{e^{-\lambda_n kt}}{\lambda_n^2 k}]\sin\frac{n\pi x}{l},$$

where $\lambda_n = (n\pi/l)^2$.