Chapter 1. Introduction

Consider the following systems (Balance laws)

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{m} \frac{\partial}{\partial x_i} f_i(u) = \varepsilon \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left(B_{ij}\,(u)\,\partial_{x_j}\,u \right) + F(u, x, t), \quad (1.1)$$

$x \,\epsilon\, R^n, \quad t > 0.$

Here $u \,\epsilon\, R^n$, $f_i(u) \,\epsilon\, R^n$, $B_{ij}(u)$ is a $n \times n$ matrix, $F \,\epsilon\, R^n (n \geq 1)$, $\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} (B_{ij}(u)\,\partial_{x_j})$ denotes an elliptic operator, and $\varepsilon \geq 0$. As is well-known, there are many physical models holding the forms of (1.1), such as MHD, viscoelasticity, relativity, liquid-crystal models and Hamilton-Jacobi systems and so on. Particularly, the compressible Navier-stokes systems are one of well-known (1.1) -type models, which are

$$
\begin{cases}
\partial_t \rho + div(\rho \vec{u}) = 0, \\
\text{(Conservation of Mass)} \\
\\
\partial_t(\rho \vec{u}) + div(\rho \vec{u} \otimes \vec{u}) + \bigtriangledown p = div\, \theta + \rho \vec{F}, \\
\text{(Conservation of Momentum)} \\
\\
\partial_t(\rho E) + div(\rho \vec{u} E + \vec{u} p) = div(\vec{u}\theta) + k \bigtriangleup T + \rho \vec{F} \cdot \vec{u} \\
\text{(Conservation of Energy)}
\end{cases}
\tag{1.2}
$$

where

$$
\begin{aligned}
\rho: &\quad \text{density}, \quad 0 \le \rho \epsilon R^1, \\
\vec{u}: &\quad \text{velocity}, \quad \vec{u} \epsilon R^d, \\
p: &\quad \text{pressure}, \quad 0 \le p \epsilon R^1, \\
\vec{F}: &\quad \text{exterior force}, \\
E: &\quad \text{total energy}, E = \frac{1}{2}|\vec{u}|^2 + e, e \text{ is the internal energy}, \\
\theta = &\quad \mu(\bigtriangledown \vec{u} + (\bigtriangledown \vec{u})^t) + \mu'(div\, \vec{u})I, \quad \mu \ge 0, \quad \mu + \frac{2}{m}\mu' \ge 0, \\
T: &\quad \text{temperature}, k: \text{heat conductive coefficient}, k \ge 0.
\end{aligned}
$$

Specially, when $\vec{F} = 0, \quad \mu = 0 = \mu', \quad k = 0$, equations (1.2) become the following Euler system for ideal compressible fluids.

$$\begin{cases} \partial_t \rho + div(\rho \vec{u}) = 0, \\ \partial_t(\rho \vec{u}) + div(\rho \vec{u} \otimes \vec{u}) + \bigtriangledown p = 0, \\ \partial_t(\rho E) + div(\rho \vec{u} E + \vec{u} p) = 0, \end{cases} \tag{1.3}$$

where $p = p(\rho, e)$. And the study on the Euler system is the driving force for the mathematical theory of shock waves. Now we give the definition of first order hyperbolic systems.

Definition 1.1 Suppose that $\varepsilon = 0$. Then the systems (1.1) is called hyperbolic if $\sum_{i=1}^{m} \xi_i \bigtriangledown_u f_i(u)$, which is an n x n matrix, has n real eigenvalues

$$\lambda_1(u,\xi) \leq \lambda_2(u,\xi) \leq \cdots \leq \lambda_n(u,\xi)$$

for any $\xi \epsilon R^n \backslash \{0\}$.

For example, the Euler equations (1.3) is hyperbolic for gas dynamics. To systems (1.1), some of the main interests are:

1. Well-posedness. As usual, it includes the existence, uniqueness, regularity of the solutions to (1.1), and it also includes the continuous dependence of the solutions.
2. Asymptotic behavior of solutions. This topic mainly includes two interesting aspects. One is about whether the solutions of Navier-Stokes equations tend to those of Euler equations as $\varepsilon \to 0$, especially in bounded domain case, which is related to boundary layer theory and internal layers, such as shock layers theory. The other aspect is about the large time behavior of the solutions.
3. Numerical methods for the weak solutions. There are two main techniques, front tracking and shock capturing, to simulate the "sharp front".

One of the most illustrating examples is the Burgers' equation

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0, \\ u(x, t = 0) = u_0(x), \end{cases} \tag{1.4}$$

where $u_0(x)$ is a given initial data. Recall the linear equation

$$\begin{cases} \partial_t u + c\,\partial_x u = 0, \\ u(x, t = 0) = \varphi(x), \end{cases} \tag{1.5}$$

where $\varphi(x)$ is a given initial data and c is a constant. The solution to (1.5) is the travelling wave solution $u = \varphi(x - ct)$. The speed of the wave is a constant ($\equiv c$). And the regularity of the solution is same as the initial function $\varphi(x)$. However, to the Burgers' equation (1.4), the speed of wave depends on itself ($= u(x,t)$). This property is essential due to the nonlinear term $u\, u_x$. Roughly speaking, the convection term has both "bad effects" and "good effects". The bad effects, say, are the singularity or shock waves. And there is no global classical solution generally no matter how smooth the initial data is. The good effects are the smoothing effects in some sense. Recall that $u(x,t)$ is only continuous to linear equation (1.5). Also, if $u_0(x) \in C_0^\infty(R^1)$, then $|u(x,t)| \to 0$ as $t \to +\infty$, which was shown by E. Hopf (1950). This is also very different from the linear case.

Now we give some details concerning important issues for the Burgers' equation (1.4). These issues are so important that one can find some traces and ideas while studying the general conservation laws systems.

§1.1 Formation of singularities

Assume that $u_0(x) \in C_B^\infty(R^1)$, which is a smooth and bounded function. Let $p = \partial_x u$. Then the equation (1.4) becomes

$$\partial_t p + u \partial_x p + p^2 = 0 \qquad (1.6)$$

Consider the characteristic curve $x = x(t)$ defined by

$$\frac{dx}{dt} = u(x, t), \quad x(t = 0) = x_0.$$

Then, along the characteristic curve, the equation (1.6) is

$$\frac{d}{dt}\, p + p^2 = 0, \tag{1.7}$$

with the initial condition

$$p(t = 0) = \partial_x\, u_0(x_0), \tag{1.8}$$

where p(t)=p(x(t),t).

Solve (1.7), (1.8) to get

$$p(t) = \frac{\partial_x u_0(x_0)}{1 + \partial_x u_0(x_0) t},$$

that is

$$\partial_x u(x(t), t) = p(x(t), t) = \frac{\partial_x u_0(x_0)}{1 + \partial_x u_0(x_0) t}. \tag{1.9}$$

Thus if there exists x_0 such that $\partial_x u_0(x_0) < 0$, then as $t \to -\frac{1}{\partial_x u_0(x_0)}$, it holds that

$$\partial_x u(x(t), t) \to -\infty.$$

So $|\partial_x u| \to +\infty \quad \text{as} \quad t \to \frac{1}{|\partial_x u_0(x_0)|}$.

If $\partial_x u_0(x) \geq 0$, then we can get the global existence for Burgers' equation (1.4) from (1.9).

This has shown that, to the Burgers' equation (1.4), $\partial_x u$ will blow-up in finite time no matter how smooth the initial data is unless the initial data is increasing. For the blow-up of solution of $n \times n$ systems, we can see F. John's paper (1974). It is also shown that, comparing with elliptic and parabolic equation(s), "weak solution", as defined as follows, is a matter of life to hyperbolic equation(s).

Definition 1.2 A bounded measurable function $u = u(x, t)$ is said to be a weak solution to the Burgers' equation (1.4), if

$$\iint \left[\partial_t \phi \, u + \partial_x \phi \, \frac{u^2}{2} \right] dx \, dt = 0, \quad \forall \phi \in C_0^\infty \left(R^1 \times R_1^+ \right). \quad (1.10)$$

§1.2 Rankine-Hugoniot jump condition

Theorem 1.1 If $u = u(x, t)$ is a piecewise smooth function with its discontinuity lying on a Lipschitz continuous curve $x = x(t)$, then it is a weak solution to the Burgers' equation (1.4) iff $u = u(x, t)$ satisfies the equation away from $x = x(t)$ pointwises and on $x = x(t)$

$$\dot{x}(t) = \frac{1}{2}\left[u\left(x(t)+, t\right) + u\left(x(t)-, t\right)\right], \tag{1.11}$$

where

$$u(x(t)+, t) = \lim_{x \to x(t), x > x(t)} u(x, t) \equiv u_r,$$

$$u(x(t)-, t) = \lim_{x \to x(t), x < x(t)} u(x, t) \equiv u_l.$$

In particular, (u_l, u_r, s) (u_l, u_r, s are constants) gives rise to a weak solution

$$u(x,t) = \begin{cases} u_l, & x < st, \\ u_r, & x > st \end{cases}$$

iff

$$s = \frac{1}{2}(u_l + u_r). \tag{1.12}$$

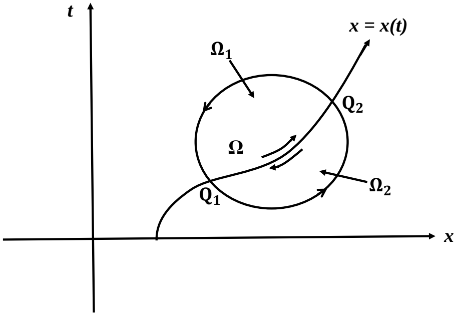

Figure 1.1

Proof of Theorem 1.1
Assume that $u = u(x, t)$ is a weak solution to Burgers equation, where $u = u(x, t)$ is a piecewise smooth function.

For any $\Omega \subset\subset R^1 \times R_1^+$ and any $\phi \epsilon C_0^\infty(\Omega)$, if $\Omega \cap \{x = x(t)\} = \phi$, it is easily concludes by divergence theorem that $u = u(x, t)$ satisfies the equation pointwise away from $x = x(t)$.

If $\Omega \cap \{x = x(t)\} \neq \phi$, (see Figure 1.1), then Ω is divided into two components Ω_1 and Ω_2 by $\Gamma : x = x(t)$. From the assumption, u is smooth in Ω_1 and Ω_2. Let $\phi \epsilon C_0^\infty(\Omega)$. By the definition of weak solutions and the divergence theorem, one has

$$
\begin{aligned}
0 &= \iint_{\Omega} \left(u\,\phi_t + \frac{u^2}{2}\phi_x \right) dx\,dt \\
&= \iint_{\Omega_1} \left(u\,\phi_t + \frac{u^2}{2}\phi_x \right) dx\,dt + \iint_{\Omega_2} \left(u\,\phi_t + \frac{u^2}{2}\phi_x \right) dx\,dt \\
&= \iint_{\Omega_1} \left[(u\,\phi)_t + \left(\frac{u^2}{2}\phi \right)_x \right] dx\,dt + \iint_{\Omega_2} \left[(u\,\phi)_t + \left(\frac{u^2}{2}\phi \right)_x \right] dx\,dt \\
&= \int_{\partial\,\Omega_1} \phi \left(-u\,dx + \frac{u^2}{2}dt \right) + \int_{\partial\,\Omega_2} \phi \left(-u\,dx + \frac{u^2}{2}dt \right)
\end{aligned}
$$

Since $\phi = 0$ on $\partial\Omega$, the above line integrals are nonzero only along Γ. Thus

$$\begin{aligned}
&\int_{\partial\,\Omega_1} \phi\left(-u\,dx + \frac{u^2}{2}dt\right) \\
= &\int_{Q_1}^{Q_2} \phi\left(-u\left(x(t)-,t\right)dx + \frac{u^2\left(x(t)-,t\right)}{2}dt\right) \\
= &\int_{Q_1}^{Q_2} \phi\left(-u\left(x(t)-,t\right)\frac{dx}{dt} + \frac{u^2\left(x(t)-,t\right)}{2}\right)dt
\end{aligned}$$

$$
\begin{aligned}
&\int_{\partial\,\Omega_2} \phi \left(-u\, dx + \frac{u^2}{2} dt \right) \\
= \quad & -\int_{Q_1}^{Q_2} \phi \left(-u\left(x(t)+, t\right) dx + \frac{u^2\left(x(t)+, t\right)}{2} dt \right) \\
= \quad & -\int_{Q_1}^{Q_2} \phi \left(-u\left(x(t)+, t\right) \frac{dx}{dt} + \frac{u^2\left(x(t)+, t\right)}{2} \right) dt
\end{aligned}
$$

Since ϕ is arbitrary, one can conclude that

$$
\frac{dx}{dt} = \frac{1}{2}\left[u\left(x(t)+, t\right) + u\left(x(t)-, t\right)\right]
$$

A similar argument gives the converse of the proof. That is , if $u = u(x,t)$ is a piecewise smooth function with its discontinuity lying on $\Gamma : x = x(t)$, and it satisfies the equation away from $x = x(t)$ pointwises and on $x = x(t)$ the relation (1.11) is satisfied, then $u = u(x,t)$ is a weak solution.

The proof of Theorem 1.1 is finished.

Remark: Relations (1.11) and (1.12) are called Rankine-Hugoniot jump conditions. The discontinuity of the weak solution $u(x,t)$ along $x = x(t)$ with $u_l > u_r$ is called a shock wave of (1.4). Theorem 1.1 shows that (u_l, u_r, s) is a shock wave (see Figure 1.2)

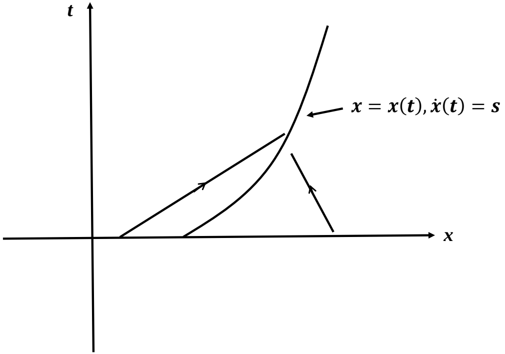

Figure 1.2

iff

$$s = \frac{1}{2}\left[u_l + u_r\right]$$

or

$$s[u] = \left[\frac{1}{2}u^2\right]$$

where

$$[v] = v(x(t)+) - v(x(t)-)$$

$s = \dot{x}(t)$ is called the speed of the shock wave.

§1.3 Loss of Uniqueness

Another feature of the Burgers equation is the loss of uniqueness of weak solutions. This can be clearly shown by applying Theorem 1.1 that $u_1(x,t) \equiv 0$ and

$$u_2(x,t) = \begin{cases} 0, & x < -\frac{\alpha}{2}t, \\ -\alpha, & -\frac{\alpha}{2}t < x < 0, \\ \alpha, & 0 < x < \frac{\alpha}{2}t, \\ 0, & x > \frac{\alpha}{2}t, \end{cases}$$

(See Figure 1.3)

are both weak solution of Burgers equation (1.4) with $u_0(x) = 0$ for any $0 < \alpha \leq 1$.

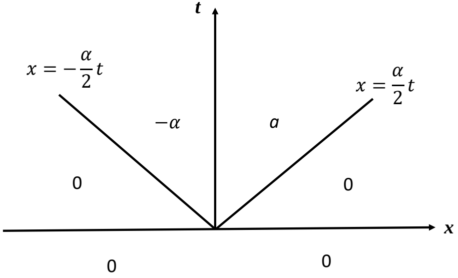

Figure 1.3

Another example is given in the following. Let

$$u_0(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Then,

$$u_3(x,t) = \begin{cases} -1, & x < 0, \\ 1, & x > 0, \end{cases}$$

and

$$u_4(x,t) = \begin{cases} -1, & x < -t, \\ \frac{x}{t}, & -t < x < t, \\ 1, & x > t, \end{cases}$$

are both weak solutions of the Burgers equation. It is interesting to note that although $u_0(x)$ is discontinuous, $u_4(x,t)$ is a continuous solution. This is again a distinct feature of nonlinear equations. In the following sections, we will give some reasonable conditions to pick out the real physical solution from so many weak solutions. Then we will see that $u_4(x,t)$ is a physical solution, which is called the rarefaction wave solution. Physical solution is unique.

§1.4 Invalidity of Nonlinear Transformations

For smooth solutions, one can take any transformation to the equation and the solution to the transformed equation is unchanged. When one deals with weak solutions, one cannot change the dependent variables in general. For example, let u be a weak solution to the Burgers equation:

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0 \\ u(x, t = 0) = u_0(x) \end{cases} \tag{1.13}$$

and consider another problem

$$\begin{cases} \partial_t(\frac{u^2}{2}) + \partial_x(\frac{u^3}{3}) = 0 \\ u(x, t = 0) = u_0(x) \end{cases} \tag{1.14}$$

they have the same smooth solutions by multiplying u in equation (1.13) to get equation (1.14). But this fails for weak solutions. For example, (u_-, u_+, s) is a shock wave for (1.13) if and only if $s = \frac{1}{2}(u_+ + u_-)$. However, (u_-, u_+, s) is a shock wave for (1.14) if and only if $(\frac{1}{2}\, u_+^2 - \frac{1}{2}\, u_-^2) \cdot s = \frac{1}{3}\, u_+^3 - \frac{1}{3}\, u_-^3$, and s cannot satisfy both of them in general. It shows that the shock speeds are not the same for (1.13) and (1.14).

§1.5 Existence of Weak Solutions to the Cauchy Problem for the Burgers Equation

Consider the Cauchy problem for the Burgers equation

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0 \\ u(x, t = 0) = u_0(x) \epsilon L^\infty(R) \end{cases} \tag{1.15}$$

There are several ways to prove the existence theorem. Here we use an intuitive physical method to find the weak solution, more precisely, by adding diffusion term to the Burgers equation and regularizing initial data, that is so called viscous problem, to get the approximate solution u^ε which is smooth, and converges to the weak solution of (1.15) in some topology which is stronger than weak topology and weaker than the c^1 topology. To do this, we need some a priori estimate on u^ε. The main theorem can be stated as follows.

Theorem 1.2
Assume that $u_0(x) \in L^\infty(R)$. Then there exists a weak solution u to the Cauchy problem (1.15). Furthermore, u can be realized as a limit of the corresponding "viscous problem"

$$\begin{cases} \partial_t u^\varepsilon + \partial_x \left(\frac{(u^\varepsilon)^2}{2} \right) = \varepsilon \, \partial_x^2 u^\varepsilon \\ u^\varepsilon(x, t = 0) = u_0^\varepsilon \end{cases} \tag{1.16}$$

where u_0^ε can be chosen as any smooth approximation of u_0.

Proof of Theorem 1.2:
First for the problem (1.16), there exists a unique smooth solution u^ε for any given $\varepsilon > 0$, and by maximal principle, $u^\varepsilon \leq M = \max\, u_0^\varepsilon$ provided that $\max\, u_0^\varepsilon \leq \max\, u_0$.

To prove the theorem, we need to prove the convergency of u^ε in some sense stronger than the one in the sense of distribution because if $u^\varepsilon \to u$ in the sense of distribution, one cannot imply that $\frac{1}{2}\,(u^\varepsilon)^2 \to \frac{1}{2}\,u^2$ in the sense of distribution. We will estimate the total variation of u^ε. Before getting this estimate, we need the one sided derivative estimate.

Let $p = \partial_x u^\varepsilon$. Since u^ε is smooth, applying ∂_x to the Burgers equation (1.15) yields

$$\begin{cases} \partial_x p + u^\varepsilon \partial_x p + p^2 = \varepsilon \partial_x^2 p \\ p(x, t = 0) = p_0(x) = \partial_x u_0^\varepsilon \end{cases}$$

We claim that $p(x, t) \leq \frac{1}{t}$ for all $x \epsilon R, \quad t > 0$. Let $Q(t)$ satisfy the following ODE:

$$\begin{cases} \frac{d}{dt} Q + Q^2 = 0 \\ Q(t = 0) = Q_0 = \max\{0, P_0\} \end{cases}$$

Homework 1: Prove the existence of weak solution using the Lax-Friedrich's scheme.

By the Comparison Principle for the operator $Dg = \partial_t g + u^{\varepsilon} \partial_x g + g^2 - \varepsilon \partial_x^2 g$ and comparing initial data, we have $p(x,t) \leq Q(t) = \frac{1}{t+\frac{1}{Q(o)}} \leq \frac{1}{t}$ for any $x \epsilon R, \quad t > 0$. Hence the claim holds.

Second, we prove that u^{ε} has local uniformly bounded total variation, that is, for $R > 0$,
$t > 0$, there is a constant $C(M, R, t)$ depending only on M, R and t such that $\int_{-R}^{R} |\partial_x u^{\varepsilon}| dx \leq C(M, R, t)$ for all $\varepsilon > 0$. Let the increasing total variation

$$ITV_{[-R,R]}\, u^\varepsilon = \int_{[-R,R]\cap\{x:\partial_x\, u^\varepsilon(x,t)>0\}} \partial_x\, u^\varepsilon\, dx$$

and the decreasing total variation

$$DTV_{[-R,R]}\, u^\varepsilon = -\int_{[-R,R]\cap\{x:\partial_x\, u^\varepsilon(x,t)<0\}} \partial_x\, u^\varepsilon\, dx.$$

Then

$$ITV_{[-R,R]}\, u^\varepsilon - DTV_{[-R,R]}\, u^\varepsilon = \int_{-R}^{R} \partial_x\, u^\varepsilon\, dx = u^\varepsilon(R,t) - u^\varepsilon(-R,t)$$

$$\left| ITV_{[-R,R]}\, u^\varepsilon - DTV_{[-R,R]}\, u \right| \le 2M \quad \forall R, t, \varepsilon > 0$$

Also $0 \leq ITV_{[-R,R]}\, u^\varepsilon \leq \int_{[-R,R]} \frac{1}{t}\, dx = \frac{2R}{t}$. Hence

$$\begin{aligned} DTV_{[-R,R]}\, u^\varepsilon &= u^\varepsilon(-R,t) - u^\varepsilon(R,t) + ITV_{[-R,R]}\, u^\varepsilon \\ &\leq 2M + \frac{2R}{t} \end{aligned}$$

It follows that

$$\begin{aligned} TV_{[-R,R]}\, u^\varepsilon &= ITV_{[-R,R]}\, u^\varepsilon + DTV_{[-R,R]}\, u^\varepsilon \\ &\leq 2M + \frac{4R}{t} \end{aligned}$$

Let $C(M,R,t) = 2M + \frac{4R}{t}$. Then the total variation of u^ε on $[-R,R]$ is bounded for each time $t > 0$.

By Helley principle, for each fixed $t > 0$, there exists a convergent subsequence of $u^{\varepsilon}(x,t)$ to $u(x,t) \quad x - a.e.$ Applying the diagonal process, there is a further subsequence, say u^{ε_i}, such that $u^{\varepsilon_i}(x,t_j) \to u(x,t_j)$ as $\varepsilon_i \to 0$ for a.e. x for all t_j, where $\{t_j\}$ be chosen as a countable dense subset of (0,T). To show that $u^{\varepsilon_i}(x,t) \to u(x,t)$ for a.e. x and for every $t\,\epsilon(0,t)$ as $\varepsilon_i \to 0$, we need to show

$$\int |u^{\varepsilon}(x,t) - u^{\varepsilon}(x,s)|dx \leq C(|t-s| + \varepsilon)$$

Let $v^\varepsilon(x,t)$ be the solution to (1.16) with initial data $v^\varepsilon(x,t=0) = u_0 * \rho_\varepsilon$, where $\rho(x)$ is the mollifier satisfying $\rho \,\epsilon\, C_0^\infty(R), \quad \rho(x) \geq 0, \quad \int_R \rho(x)dx = 1, \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$. Then for $0 < s < t$, one has

$$\begin{aligned}
& ||u^\varepsilon(\cdot,t) - u^\varepsilon(\cdot,s)||_{L^1} \\
\leq\ & ||u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)||_{L^1} + ||v^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,s)||_{L^1} + ||v^\varepsilon(\cdot,s) - u^\varepsilon(\cdot,s)||_{L^1} \\
\leq\ & ||u_0^\varepsilon - u_0 * \rho_\varepsilon||_{L^1} + ||v^\varepsilon(\cdot,t-s) - v_0^\varepsilon||_{L^1} + ||u_0^\varepsilon - u_0 * \rho_\varepsilon||_{L^1} \\
\leq\ & 2\,||u_0||_{BV} \cdot \varepsilon + 2\,||u_0^\varepsilon - u_0||_{L^1} + \int_0^{t-s} ||v_t^\varepsilon||_{L^1}\, dt
\end{aligned}$$

Now we claim that $||v_t^\varepsilon(\cdot,\tau)||_{L^1} \leq C$ for any ε and τ. Let $w^\varepsilon = \partial_t v^\varepsilon$. Applying ∂_t to (1.15) yields

$$\begin{cases} \partial_t w^\varepsilon + \partial_x(v^\varepsilon w^\varepsilon) = \varepsilon \partial_x^2 w^\varepsilon \\ w^\varepsilon(x, t=0) = \varepsilon v_{0xx}^\varepsilon - v_0^\varepsilon v_{0x}^\varepsilon = w_0^\varepsilon \end{cases}$$

Then

$$\begin{aligned} ||w^\varepsilon||_{L^1} &\leq ||w_0^\varepsilon||_{L^1} \\ &\leq \varepsilon ||v_{0xx}^\varepsilon||_{L^1} + M ||v_{0x}^\varepsilon||_{L^1} \\ &= \varepsilon ||u_0^{''} * \rho_\varepsilon||_{L^1} + M ||u_0^{'} * \rho_\varepsilon||_{L^1} \\ &\leq \varepsilon ||u_0^{'} * \rho_\varepsilon^{'}||_{L^1} + M ||u_0^{'}||_{L^1} \cdot ||\rho_\varepsilon||_{L^1} \\ &\leq ||u_0^{'}||_{L^1} \cdot \varepsilon ||\rho_\varepsilon^{'}||_{L^1} + M ||u_0^{'}||_{L^1} \\ &\leq C(M) ||u_0||_{BV} \end{aligned}$$

Hence $||u^\varepsilon(\cdot,t) - u^\varepsilon(\cdot,s)||_{L^1} \leq C(|t-s| + \varepsilon)$ for all $0 < s < t$, $\varepsilon > 0$, where C_1 depends only on M, $||u||_{BV}$.

It follows from the above time estimate that $u^{\varepsilon_i}(x,t) \to u(x,t)$ for a.e. x and for every $t \,\epsilon (0,T)$ as $\varepsilon_i \to 0$. Hence for any $\varphi \,\epsilon\, C_0^\infty(R \times R_+)$, it holds that as $\varepsilon_i \to 0$,

$$\iint \partial_t \varphi \cdot u^{\varepsilon_i} \, dx \, dt \to \iint \partial_t \varphi \cdot u \, dx \, dt$$

$$\iint \partial_x \varphi \frac{(u^{\varepsilon_i})^2}{2} \, dx \, dt \to \iint \partial_x \varphi \cdot \frac{u^2}{2} \, dx \, dt$$

$$\varepsilon_i \iint \partial_x^2 \varphi \cdot u^{\varepsilon_i} \, dx \, dt \to 0$$

It shows that u must be a weak solution to the Burgers equation.

This proves the theorem.

Remark 1: In fact, we can further show that u^ε is a Cauchy sequence in the sense that $I_{\varepsilon_1,\varepsilon_2}(t) = \int |u^{\varepsilon_1}(x,t) - u^{\varepsilon_2}(x,t)|dx \to 0$ as $\varepsilon_1, \varepsilon_2 \to 0$ by the fact $I_{\varepsilon_1,\varepsilon_2}(t_j) \to 0$ as $\varepsilon_1, \varepsilon_2 \to 0$ for all t_j as stated in the proof of the theorem and the time estimate. Therefore $u^\varepsilon(x,t) \to u(x,t)$ as $\varepsilon \to 0$ for a.e. x and all t.

Remark 2: For any $\varphi \epsilon C_0^\infty(R \times R_+)$ with $\varphi \geq 0$,

$$-\iint \partial_x \varphi \cdot u^\varepsilon \, dx\, dt = \iint \varphi \cdot \partial_x u^\varepsilon \, dx\, dt \leq \frac{1}{t} \iint \varphi \, dx\, dt$$

Taking $\varepsilon \to 0^+$, we obtain $-\iint \partial_x \varphi \cdot u \, dx\, dt \leq \frac{1}{t} \iint \varphi \, dx\, dt$.

So the weak solution satisfies the entropy condition

$$\partial_x u \leq \frac{1}{t} \text{ in the sense of distribution} \qquad \text{(E)}$$

which is called Oleinik entropy condition.

§1.6 Entropy Condition

Because of the nonuniqueness of weak solutions to the Burgers equation, one needs we want to find the physically meaningful solutions. The best description to those solution is to introduce the entropy condition. First we state the basic property to the entropy weak solutions which have shocks.

Fact: (u_-, u_+, s) is a weak solution to the Burgers equation and satisfies (E) if and only if $u_+ < s < u_-$. This condition is called Lax geometric entropy condition.

Proof: Since (u_-, u_+, s) is a weak solution, so $s = \frac{1}{2}(u_+ + u_-)$, and by the entropy condition, one must jump down across the discontinuity, $u_- > u_+$. Then $u_+ < s = \frac{1}{2}(u_+ + u_-) < u_-$.

Conversely, if $u_+ < s < u_-$, then $\partial_x u \leq 0 \leq \frac{1}{t}$.

Consider

$$\begin{cases} \partial_x u + u \, \partial_x u = 0 \\ u(x, t = 0) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0 \end{cases} \end{cases}$$

and its entropy weak solution, then $u_+ < s < u_-$. We look at the characteristic curve on the left and right of the shock. The characteristic speed $\frac{dx}{dt} = u$.

On the left of the shock, the characteristic curve is $x = x_0 + u_- t$ with $x_0 < 0$. Since $u_- > s$, the characteristic curve will intersect with shock curve $x = s \cdot t$ in finite time. Similarly, on the right of the shock, the characteristic curve $x = x_0 + u_+ t$, $x_0 > 0$, will intersect with shock curve in finite time. This fact gives a geometric meaning of the compressibility condition of a shock, that is, all characteristic curve on both sides will be absorbed by the shock curve (see Figure 1.2).

It should be noted that for no-entropy jumps, the characteristic lines run away from the jump, see Figure 1.4 below.

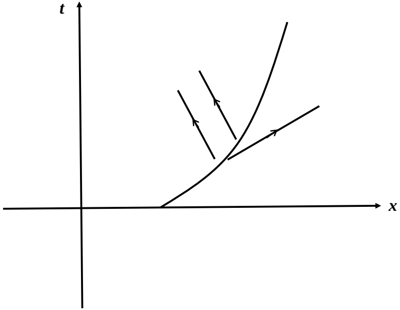

Figure 1.4

The other important fact for introducing the entropy condition is that the Lax entropy condition is a sufficient and necessary condition of structural stability of shock wave.

Let $(u_l(x,t), u_r(x,t), \sigma(t))$ be a perturbed shock of (u_-, u_+, s). $\sigma(t) = \dot{x}(t) = s + \delta_\triangle$, then the perturbed solution satisfies

$$
\begin{array}{lll}
\partial_t u_l + u_l \, \partial_x u_l = 0 & \text{on} & \Omega_l = \{(x,t) \quad | \quad x < x(t)\} \\
\partial_t u_r + u_r \, \partial_x u_r = 0 & \text{on} & \Omega_r = \{(x,t) \quad | \quad x > x(t)\}
\end{array}
$$

Define new variable $X = x - x(t)$. Then it changes to

$$
\begin{array}{lll}
\partial_t u_l - \sigma(t) \, \partial_X u_l + u_l \, \partial_X u_l = 0 & \text{on} & \Omega_l = \{(X,t) \,|\, X < 0\} \\
\partial_t u_r - \sigma(t) \, \partial_X u_r + u_r \, \partial_X u_r = 0 & \text{on} & \Omega_r = \{(X,t) \,|\, X > 0\}
\end{array}
$$

Write $u_l = u_- + \delta_l, \quad u_r = u_+ + \delta_r, \quad \sigma = s + \delta_\triangle$. One can compute the linearized equation by simply dropping the quadratic terms of $\delta_l, \delta_r, \delta_\triangle$ to get

$$\begin{array}{lll} \partial_t \delta_l + (u_- - s)\partial_X \delta_l = 0 & \text{on} & \Omega_l = \{(X,t) \,|\, X < 0\} \\ \partial_t \delta_r + (u_+ - s)\partial_X \delta_r = 0 & on & \Omega_r = \{(X,t) \,|\, X > 0\} \end{array}$$

Now we determine the condition of stability of the linearized equation of δ_l, δ_r. If $u_- - s > 0$, then the characteristic line from the left intersect the shock curve $X = 0$, the problem is well-posed. If $u_- - s < 0$, the problem is ill-posed. If $u_+ - s > 0$, then the characteristic curve from the right will not intersect the shock curve at any time $t > 0$, so the problem is ill-posed. If $u_+ - s < 0$, then the problem is well-posed. Hence the linearized problem is stable if and only if $u_+ < s < u_-$.

§1.7 Uniqueness of Weak entropy solutions

In an example given before, there are more than one weak solutions to the Cauchy problem. Now one can apply the entropy condition to determine the admissible discontinuity and select only one of them. We have also shown the equivalence of the two entropy conditions. In this section, we will give a rigorous proof that the entropy condition indeed ensures the uniqueness of the weak entropy solutions.

Theorem 1.3

Let u_1 and u_2 be two weak solutions of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0 \\ u(x, t = 0) = u_0(x) \end{cases}$$

with the same initial data. Assume further that both u_1, u_2 satisfy Oleinik entropy condition:

$$\partial_x u_i \leq \frac{1}{t}, \quad i = 1, 2.$$

Then

$$u_1 \equiv u_2, \quad a.e.$$

Proof: (Potential Method) We give the proof in two steps.
Step 1: To reduce the uniqueness of weak solutions to a nonlinear problem to a solution to a linear problem:

$$\partial_t(u_i) + \partial_x(\frac{1}{2}\, u_i^2) = 0, \quad i = 1, 2, \quad \text{i.e.} \quad div(u_i, \frac{1}{2}\, u_i^2) = 0$$

Then by the Green formula,

$$\varphi_i(x,t) = \int_{(0,0)}^{(x,t)} (-u_i\, dx + \frac{1}{2}\, u_i^2\, dt)$$

is well-defined and independent of the path one chooses, and

$$\partial_x\, \phi_i = -u_i, \quad \partial_t\, \phi_i = \frac{1}{2}\, u_i^2$$

Thus to show $u_1 \equiv u_2$, a.e. it suffices to show that $\varphi_1 \equiv \varphi_2$, a.e.. Set $\varphi = \varphi_2 - \varphi_1$. Then

$$\begin{aligned} \partial_t \varphi = \partial_t \varphi_2 - \partial_t \varphi_1 = \frac{1}{2} u_2^2 - \frac{1}{2} u_1^2 &= \frac{1}{2} (u_1 + u_2)(u_2 - u_1) \\ &= -\frac{1}{2}(u_1 + u_2)\, \partial_x \varphi, \end{aligned}$$

i.e.

$$\left\{ \begin{array}{rl} \partial_t \varphi + a\, \partial_x \varphi = 0, & \text{with } a = \frac{1}{2}(u_1 + u_2), \\ \varphi(x, t = 0) & = \varphi_2(x, 0) - \varphi_1(x, 0) \\ & = \int_0^x (-u_2(y, 0))dy - \int_0^x (-u_1(y, 0))dy = 0. \end{array} \right. \tag{1.17}$$

Step 2: (1.17) has only one solution $\varphi \equiv 0$.
If a is continuous, then along the particle path $x = x(t)$,

$$\frac{dx(t)}{dt} = a(x(t), t), \quad \varphi(x(t), t) = \varphi(x(0), 0) = 0$$

Unfortunately, a is not smooth in general.

Multiplying (1.17) by φ and integrating with respect to x from $-\infty$ to $+\infty$ show that

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \varphi^2(x,t)dx + \int_{-\infty}^{+\infty} a\,\partial_x\,\varphi^2(x,t)dx = 0$$

Claim:

$$-\int_{-\infty}^{+\infty} a\,\partial_x\,\varphi^2\,dx = \int_{-\infty}^{+\infty} \frac{\partial a}{\partial x}\,\varphi^2\,dx \tag{1.18}$$

Then by the claim, and noting that $\frac{\partial a}{\partial x} = \frac{1}{2}(\partial_x\,u_1 + \partial_x\,u_2) \leq \frac{1}{t}$, one has

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \varphi^2(x,t)dt = \int_{-\infty}^{+\infty} \frac{\partial a}{\partial x}\varphi^2\,dx \leq \frac{1}{t}\int_{-\infty}^{+\infty} \varphi^2\,dx,$$

i.e.

$$\frac{d}{dt}\,||\,\varphi(\cdot,t)||_{L^2}^2 \leq \frac{1}{t}\,||\,\varphi(\cdot,t)||_{L^2}^2.$$

Now, $\forall\, 0 < s < t$, integrate the above differential inequality,

$$||\varphi(\cdot,t)||_{L^2}^2 \leq \left(\frac{t}{s}\right) ||\varphi(\cdot,s)||_{L^2}^2,$$

i.e.

$$||\varphi(\cdot,t)||_{L^2} \leq \left(\frac{t}{s}\right)^{\frac{1}{2}} ||\varphi(\cdot,s)||_{L^2}.$$

To prove $||\varphi(\cdot,t)||_{L^2} = 0, \quad \forall \quad t > 0$, it suffices to show $||\varphi(s)||_{L^2} = O(s)$.

To determine $\varphi_i(x,s)$, one can choose the path that consists of the lines connecting $(0,0)$ to $(x,0)$ and $(x,0)$ to (x,s).

Then

$$\begin{aligned}\varphi(x,s) &= \varphi_2(x,s) - \varphi_1(x,s) \\ &= \int_0^s \left(\frac{1}{2}\,u_2^2 - \frac{1}{2}\,u_1^2\right)\,dt = O(1)s,\end{aligned}$$

letting s tend to 0, we have $||\varphi(t)||_{L^2} = 0, \quad t > 0.$

Proof of the claim (1.18): Define a Friedrichs mollifier in the following way, $\forall\, \rho \,\epsilon\, C_0^\infty(R^1)$,

$$\rho \geq 0, supp\, \rho\, \epsilon (-1,1), \int_{-\infty}^{+\infty} \rho(x)\, dx = 1,$$

$$\rho_h(x) = \frac{1}{h} \rho \left(\frac{x}{h}\right). \quad \forall f, f^h = \rho_h * f.$$

Then

$$\begin{aligned} -\int a\, \partial_x\, \varphi^2\, dx &= -\int a^h\, \partial_x\, \varphi^2\, dx + \int \left(a^h - a\right) \partial_x\, \varphi^2\, dx \\ &= \int \frac{\partial a^h}{\partial x}\, \varphi^2\, dx + E_h, \end{aligned}$$

Since $E_h = \int (a^h - a) \partial_x\, \varphi^2\, dx \to 0$ as $\quad h \to 0$,
$RHS \to \int \frac{\partial a}{\partial x}\, \varphi^2\, dx$ as $\quad h \to 0$. The proof is finished.

§1.8 L^1 - Contraction principle

In this section, we will present an important property of the weak entropy solutions, which shows that the weak entropy solution not only is stable in $L^1(R)$ but also has some contraction property which will be made clear in the following theorem. Thus, the Cauchy problem is well-posed in the class of entropy weak solutions in $L^1(R)$ for the Burgers equation. In the proof we adopt the vanishing viscosity method. And we will see that the contraction property is due essentially to the maximum principle of the equation.

Theorem 1.4 Let u and v be two entropy weak solutions to the Cauchy problem for the Burgers equation with initial data $u_0(x), v_0(x)$, respectively, then

$$\int_R |u(x,t) - v(x,t)| dx \le \int_R |u_0(x) - v_0(x)| dx \qquad (1.19)$$

In fact, $\int_{|x|\le R} |u(x,t) - v(x,t)|\, dx \le \int_{|x|\le R+Mt} |u_0(x) - v_0(x)|\, dx, M = \max_x\{|u_0(x)|, |v_0(x)|\}$.

Proof: (Viscosity Method)
We start with the following problems and then take limit,

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = \varepsilon\, u_{xx}, \\ u(x, t=0) = u_0(x), \end{cases} \quad \begin{cases} \partial_t v + \partial_x \left(\frac{v^2}{2}\right) = \varepsilon\, v_{xx}, \\ v(x, t=0) = v_0(x). \end{cases}$$

Set $\sigma = u - v$. Then

$$\begin{cases} \partial_t \sigma + \partial_x (a\,\sigma) = \varepsilon\, \partial_x^2 \sigma \\ \sigma(x, t=0) = \sigma_0(x) = u_0(x) - v_0(x), \quad a = \frac{1}{2}(u+v), \end{cases} \tag{1.20}$$

If $u_0 \geq v_0$, then by the maximum principle we have
$u(x,t) - v(x,t) \geq 0$.

Hence $\int |u(x,t) - v(x,t)|dx = \int u(x,t) - v(x,t)dx =$
$\int u_0(x) - v_0(x)dx = \int |u_0(x) - v_0(x)|dx$, which is what we need.

In the following we consider the case when $u_0 \geq v_0$ is not true. Given $h > 0$, let $S_h(\sigma)$ be an convex approximation of $S(\sigma) = |\sigma|$, for example,

$$S_h(\sigma) = \begin{cases} -\sigma, & \sigma \leq -h, \\ -\frac{\sigma^4}{8h^3} + \frac{3\sigma^2}{4h} + \frac{3h}{8}, & -h \leq \sigma \leq h, \\ \sigma, & \sigma \geq h. \end{cases}$$

Multiply (1.20) by $S_h'(\sigma)$ to get

$$
\begin{aligned}
& \partial_t(S_h(\sigma)) + S_h'(\sigma)\,\partial_x(a\,\sigma) \\
= \quad & \varepsilon\, S_h'(\sigma)\,\partial_x^2\,\sigma \\
= \quad & \varepsilon\,\partial_x\left(S_h'(\sigma)\,\partial_x\,\sigma\right) - \varepsilon\, S_h''(\sigma)\sigma_x^2 \\
= \quad & \varepsilon\,\partial_x^2\,(S_h(\sigma)) - \varepsilon\, S_h''(\sigma)\,\sigma_x^2 \\
\leq \quad & \varepsilon\,\partial_x^2\,(S_h(\sigma))\,.
\end{aligned}
$$

And note also that

$$
\begin{aligned}
S_h^{'}(\sigma)\partial_x(a\,\sigma) &= S_h^{'}(\sigma)\cdot a\cdot\partial_x\,\sigma + S_h^{'}(\sigma)(\partial_x\,a)\cdot\sigma \\
&= a(S_h(\sigma))_x + \partial_x\,a\,S_h^{'}(\sigma)\,\sigma \\
&= a\,\partial_x(S_h(\sigma)) + (\partial_x a)\cdot S_h(\sigma) + \partial_x\,a\left(S_h^{'}(\sigma)\,\sigma - S_h(\sigma)\right) \\
&= \partial_x\,\left(a\,S_h(\sigma)\right) + E_h, \quad \text{where} \quad E_h \\
&= (\partial_x\,a)\cdot\left(S_h^{'}(\sigma)\,\sigma - S_h(\sigma)\right).
\end{aligned}
$$

Then by the definition of $S_h(\sigma)$, it is easy to see that

$$E_h \to 0, \quad as \quad h \to 0.$$

Therefore, $\partial_t\,S_h(\sigma) + \partial_x(aS_h(\sigma)) \le \varepsilon\,\partial_x^2\,(S_h(\sigma)) - E_h$.

Taking limit as $h \to 0^+$, one gets

$$\partial_t|\sigma| + \partial_x(a|\sigma|) \leq \varepsilon\, \partial_x^2(|\sigma|). \tag{1.21}$$

Integrating (1.21) with respect to x over R yields

$$\frac{d}{dt}\int_R |\sigma| dx \leq 0, \quad \text{which is needed for (1.19).}$$

To show the finite speed of propagation, let ρ_h be the Friedrichs mollifier defined before with additional constrain that

$$\rho^{'}(x) \geq 0, \quad x \geq 0, \quad \text{and} \quad \rho^{'}(x) \leq 0, \quad x \leq 0.$$

Set $\psi(x) = 1 - \int_{-\infty}^{x} \rho_h(\tau)\, d\tau$. Then multiplying (1.21) by $\psi(|x| - R + Mt - Ms + h)$ where s is fixed, t varies from 0 to s, and integrating by parts to obtain

$$\begin{aligned} & \frac{d}{dt}\int_R \psi(y)\,|\sigma|\,dx \\ \leq\ & \int_R |\sigma|\psi^{'}(y)\,y_t^{'}\,dx + \int_R a\,|\sigma|\,\psi^{'}(y)\,y_x^{'}\,dx + \varepsilon\int_R |\sigma|\,\psi^{''}(y)\,dx \\ =\ & \int_R |\sigma|(-\rho_h(y))(M + a\cdot sgn\,x)dx - \varepsilon\int_R |\sigma|\,\rho_h^{'}(y)\,dx \\ =\ & I_1 + I_2, \end{aligned}$$

where $y = |x| - R + Mt - Ms + h$.

For I_1, since $\rho_h(y) \geq 0$, $|a \cdot sgn\ x| = |\frac{u+v}{2}| \leq M$, so $I_1 \leq 0$.

For I_2,

$$\begin{aligned} |I_2| &\leq \varepsilon\, 2M \int_R |\rho_h'(y)|\, dx \\ &\leq \varepsilon \cdot 2M \cdot 2 \cdot \frac{1}{h}\, \rho(0) = 4\rho(0)\, M \cdot \frac{\varepsilon}{h} \quad . \end{aligned}$$

Take $h = \varepsilon^{\frac{1}{2}}$, then $|I_2| = O(1)\, \varepsilon^{\frac{1}{2}} \to 0$, as $\varepsilon \to 0$.

Therefore,

$$\begin{aligned} &\int_{-\infty}^{+\infty} \psi(|x| - R + h)\, |\sigma|\, (x, s)\, dx \\ \leq &\int_{-\infty}^{+\infty} \psi(|x| - R - Ms + h)\, |\sigma|\, (x, 0)\, dx + \int_0^s (I_1 + I_2)\, dt. \end{aligned}$$

Taking limit as $h = \varepsilon^{\frac{1}{2}} \to 0$, we have

$$\int_{|x|\le R} |u(x,s) - v(x,s)| dx \le \int_{|x|\le R+Ms} |u_0(x) - v_0(x)| dx.$$

This finishes the proof.

Remark (i): The contraction property not only implies the uniqueness but also shows a more precise fact that the value of u at (x,t) depends only on the restriction of u_0 to the interval $[x - Mt, x + Mt]$.

Remark (ii): In (1.19), take $v = u(x+h,t)$, for any give $h > 0$, one has

$$\int_R |u(x,t) - u(x+h,t)|dx \leq \int_R |u_0(x) - u_0(x+h)|dx.$$

Divide it by h and take $h \to 0$, then

$$\begin{array}{rcl} \int_R |\partial_x\, u(x,t)|dx & \leq & \int_R |\partial_x\, u_0(x)|dx \\ i.e. \quad TV\ \ u(\cdot,t) & \leq & TV\ \ u_0(\cdot) \quad . \end{array}$$

This shows the Cauchy problem is also well-posed in $BV(R)$. And the contraction property therefore becomes the basic principle for modern high resolution numerical schemes for conservation laws.

Remark (iii): The solution operator is Lipschitz continuous in time only L^1-norm (exercise).

§1.9 Riemann Problems

The Riemann problem is the Cauchy problem in the particular case of a given initial condition of the form

$$u_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases}$$

The role of Riemann problem is to furnish all the solutions of the Cauchy problem which are invariant under the group of homotheties $(x, t) \to (ax, at)$, a group which leaves invariant all the conservation form of the first order.

First, we note that the Burgers equation is dilation invariant, i.e.

$$\begin{cases} x \to y = ax \\ t \to \tau = at, \end{cases} \quad \text{then } \partial_\tau u + \partial_y \left(\frac{u^2}{2}\right) = 0.$$

Second, the initial data is dilation invariant. Thus, one would like to have solutions which depend only on the variable $\xi = \frac{x}{t}$.

Assume $u(x,t) = U(\frac{x}{t}) = U(\xi)$. Then

$$\begin{aligned}
&\partial_x u(x,t) = U'(\xi)\frac{\partial \xi}{\partial x} = U'(\xi)\cdot \frac{1}{t},\\
&\partial_t u(x,t) = U'(\xi)\frac{\partial \xi}{\partial t} = U'(\xi)\cdot (-\frac{1}{t}\xi),\\
&0 = \partial_t u + \partial_x(\frac{u^2}{2}) = \partial_t u + u\,\partial_x u = -\frac{1}{t}\,U'(\xi)\xi + UU'(\xi)\cdot \frac{1}{t},
\end{aligned}$$

i.e. $U'(\xi)(U(\xi)-\xi) = 0$, and

$$\left.\begin{array}{l} u_-, x<0 \\ u_+, x>0 \end{array}\right\} = \lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} U(\frac{x}{t}) = \left\{\begin{array}{l} \lim_{\xi\to-\infty} U(\xi) = u_-, x<0 \\ \lim_{\xi\to+\infty} U(\xi) = u_+, x>0. \end{array}\right.$$

Thus the problem for $u(x,t)$ reduces to the following problem for $U(\xi)$,

$$\begin{cases} U'(\xi)[U(\xi)-\xi]=0, \\ U(-\infty)=u_-, U(+\infty)=u_+ \end{cases}$$

when $u_- = u_+$, one gets the constant solution $U = u_- = u_+$.

Next we solve the problem for $u_- \neq u_+$.

Case 1: If there is a continuous solution, (u_-, u_+) cannot be arbitrary. To see this, note that from the equation one implies $U'(\xi) = 0$ or $U(\xi) = \xi$ and in either case, $\partial_\xi U(\xi) \geq 0$, which requires $u_+ > u_-$. In fact, if $u_+ > u_-$, we define

$$U(\xi) = \begin{cases} u_-, & \xi < u_-, \\ \xi, & u_- \leq \xi \leq u_+, \\ u_+, & \xi > u_+. \end{cases}$$

It can be seen that it solves the problem and $\partial_x u \leq \frac{1}{t}$. Hence it is the unique entropy solution to the Riemann problem. This solution is called a centered rarefaction wave. See Figure 1.5.

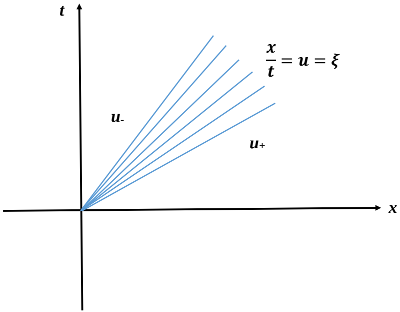

Figure 1.5

Case 2: $u_+ < u_-$. In this case, the solution cannot be continuous. Define

$$U(\xi) = \begin{cases} u_-, & \xi < s, \\ u_+, & \xi > s, \end{cases}$$

where $s = \frac{1}{2}(u_- + u_+)$. Then $\partial_x u \leq \frac{1}{t}$ in the distribution sense. And it is the unique entropy solution to the Riemann problem. This is called a shock wave solution. See Figure 1.6.

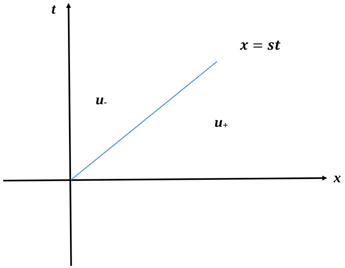

Figure 1.6

Remarks:

(i) There are three kinds of self-similar solutions, shock waves, centered rarefaction wave and constants. By uniqueness, there are no any other kind of the Riemann problem of Burgers equation. So the shock wave and rarefaction wave are basic nonlinear waves, and they are so simple because of their self-similarity.

(ii) As to the general Cauchy problem, its solutions can be characterized locally and globally by the Riemann solutions. Furthermore, we will see later that one can build up the general solutions using these basic waves.

§1.10 Wave Interactions

A. Shock wave overtakes another one

Let

$$u_0(x) = \begin{cases} u_l, & x < b, \\ u_m, & b < x < a, \\ u_r, & x > b, \end{cases}$$

where a, b are two real numbers and $b < a$. Assume that $u_l > u_m$ and $u_m > u_r$. Then, by Lax geometric entropy condition, there appear two shocks initially with speed $s_1 = \frac{1}{2}(u_m + u_r)$ and $s_2 = \frac{1}{2}(u_l + u_m)$ respectively. Since $s_2 > s_1$, two shock waves must overtake each other at the point (x^*, t^*), and when they meet, they combine to form a strong shock with speed $s_3 = \frac{1}{2}(u_l + u_r)$ (See Figure 1.7).

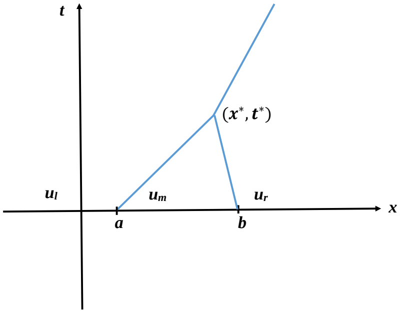

Figure 1.7

B. Shock wave overtakes a rarefaction wave

Let

$$u_0(x) = \begin{cases} u_l, & x < -1, \\ u_m, & -1 < x < 0, \\ u_r, & x > 0. \end{cases}$$

Assume that $u_l > u_m$ and $u_m < u_r$. Then there appear one shock wave with speed $s = \frac{1}{2}(u_l + u_m)$ and one rarefaction wave initially. Note that the speed of the left edge in the characteristic curve of the rarefaction wave is u_m and the speed of the shock wave is $s = \frac{1}{2}(u_l + u_m) > u_m$. So the shock wave curve $x = -1 + st$ must meet the left edge of rarefaction wave $x = u_m\, t$ at finite time $(x_*, t_*), \quad 0 < t_* < +\infty$. In fact, $x_* = \frac{2\, u_m}{u_l - u_m}, t_* = \frac{2}{u_l - u_m}$.

For $t > t_*$, there is an extension of the shock wave curve $x = \chi(t)$. For convenience, we may set $u(x(t)-, t) = u_l$. The jumping condition tells us that the speed of the shock wave is

$$\frac{d}{dt}\chi(t) = \frac{1}{2}\left(u_l + u(x(t)+, t)\right).$$

Noting that the rarefaction wave has the form

$$u(x,t) = \frac{x}{t} \quad \text{when} \quad \frac{u_m}{t} < x < \frac{u_r}{t},$$

one can get an ODE

$$\frac{d\chi(t)}{dt} = \frac{1}{2}\left(u_l + \frac{\chi(t)}{t}\right), \quad t > t_*$$

with $\chi(t = t_*) = x_*$.

Solve this equation to get

$$\chi(t) = t\, u_l + \sqrt{t}\left(\frac{\chi_* - t_* u_l}{\sqrt{t_*}}\right).$$

Substituting $\chi_* = \frac{2\,u_m}{u_l - u_m}$, $\quad t_* = \frac{2}{u_l - u_m}$ into above equality, we reduce

$$\chi(t) = t\,u_l - \sqrt{2t} \cdot \sqrt{u_l - u_m}, \quad t_* < t < t^*,$$

where we denote by t^* the end time point at which the shock wave get through the rarefaction wave. Here we do not want to give a detailed and analytical reduction by using the ODE solution. We just depict the point by separating the issue into the following cases and show it in the respective figures.

(1) If $u_l > u_r$, then $t^* < +\infty$. After t^*, there exists only one shock wave left, with speed $\tilde{s} = \frac{1}{2}(u_l + u_r)$. The shock must become weaker. (See Figure 1.8)

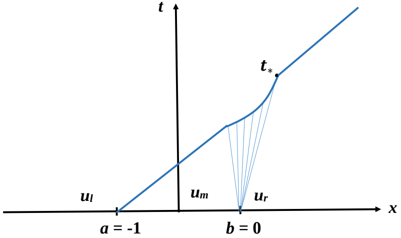

Figure 1.8

(2) If $u_l < u_r$, then $t^* = +\infty$. However, the strength of the shock goes to zero as $t \to +\infty$. And there is a centered rarefaction wave left. (See Figure 1.9)

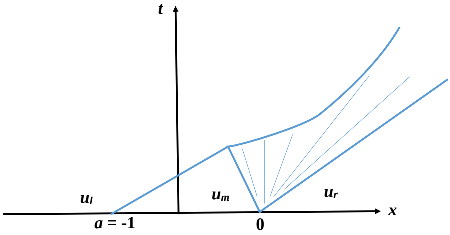

Figure 1.9

(3) If $u_l = u_r$, then $t^* = +\infty$. And there is nothing left. (See Figure 1.10)

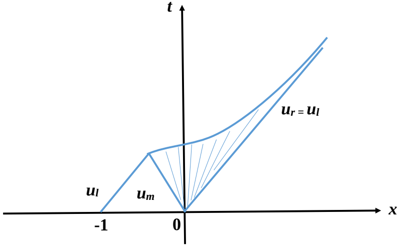

Figure 1.10

§1.11 Dissipation and time irreversibility

As we mentioned before, the linear hyperbolic equation (1.5) has a travelling wave solution $u(x,t) = \varphi(x - ct)$. We say that the linear hyperbolic equation (1.5) has no dissipation and time is reversible for the equation (1.5). In other words, the pattern of the solution of linear hyperbolic equation (1.5) is same as that of the initial data. (See Figure 1.8)

However, for the Burgers equation, there is a dissipation in a certain sense, as mentioned in the introduction. And we also have given an example in the section "Loss of Uniqueness" that the Burgers equation has continuous solutions although the initial data may be discontinuous, for example,

$$u_0(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

Actually, if $u_0(x) \epsilon L^\infty(R^1)$, we have shown that $\partial_x u(x,t) \leq \frac{1}{t}$ and $u(x,t) \epsilon BV(R^1)$ for the solution of the Burgers equation.

We also claim that the Burgers equation has time irreversibility, i.e. one can not tell what is the past for the solution of the Burgers equation.

Here is an example shown by the following figures:

After some time t_0, suppose that the solution of the Burgers equation is

$$u(x,t) = \begin{cases} 1, & x < 0 \\ -1, & x > 0 \end{cases} \tag{1.22}$$

for $t > t_0 = 1$.

But the initial state can not be determined. (See Figure 1.9)

$$u_0(x) = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases}$$

or

$$u_0(x) = \begin{cases} 1, & x < -\delta, \\ -x, & -\delta < x < \delta, \\ -1, & x > \delta. \end{cases}$$

for some $\delta \in [0, 1]$, or any other choice of the initial data.

§1.12 Large time Asymptotic Behavior (Periodic case)

Periodic Case

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0, \\ u(x, t = 0) = u_0 \, \epsilon \, L^\infty(R^1), \\ u_0(x + p) = u_0(x), \quad p \text{ is the period.} \end{cases} \tag{1.23}$$

We have the following theorem:

Theorem 1.5

(1) There exists a unique weak solution $u(x,t)$ which is space-periodic;

(2) $|u(x,t) - \bar{u}|_{L^\infty([0,p])} \leq \frac{2p}{t}$, where

$$\bar{u} = \frac{1}{p}\int_0^p u_0(x)\,dx.$$

Remark: This result is also true for general convex conservation law. Here we give a simple and direct proof for the Burgers equation.

Proof: (1) By our previous argument there exists a unique entropy weak solution $u(x,t)$ to (1.23).

Set $u_1(x,t) = u(x,t), \quad u_2(x,t) = u(x+p,t)$. Clearly, both $u_1(x,t)$ and $u_2(x,t)$ satisfy the Burgers equation and

$$\partial_x u_i \le \frac{1}{t}, \quad i = 1, 2.$$

To show $u_1 = u_2$, it suffices to show $u_1(x, t=0) = u_2(x, t=0)$, since the entropy weak solution to (1.23) is unique.

However,

$$u_1(x, t = 0) = u_0(x), \quad u_2(x, t = 0) = u(x+p, t = 0) = u_0(x+p) = u_0.$$

So,

$$u(x, t) = u(x + p, t) \quad a.e. \quad (x, t) \,\epsilon\, R^1 \times R_1^+,$$

which implies $u(x, t)$ is periodic. The first part of the theorem has been proved.

(2) Consider

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = \varepsilon\, \partial_x^2 u, \quad \varepsilon > 0. \\ u(x, t = 0) = u_0^\varepsilon. \end{cases} \tag{1.24}$$

u_0^ε is a regularization of u_0 in such a way that u_0^ε is also periodic and

$$\frac{1}{p}\int_0^p u_0^\varepsilon(x)dx = \frac{1}{p}\int_0^p u_0(x)dx = \bar{u}.$$

Then (1.24) has a unique smooth solution $u^\varepsilon(x,t)$ such that

$$\partial_x u^\varepsilon(x,t) \le \frac{1}{t}, \quad t > 0.$$

So,

$$\int_{\{x|x\,\epsilon[0,p],\, \partial_x u^\varepsilon \ge 0\}} \partial_x u^\varepsilon(x,t)dx \le \frac{p}{t}. \tag{1.25}$$

We use the following notations:

$$\begin{aligned} ITV_{[0,p]} &= \text{Increase Total Variation on } [0,p], \\ DTV_{[0,p]} &= \text{Decrease Total Variation on } [0,p], \\ TV_{[0,p]} &= \text{Total Variation on } [0,p]. \end{aligned}$$

Then (1.25) implies that

$$ITV_{[0,p]}\, u^{\varepsilon} \leq \frac{p}{t}.$$

On the other hand, by the previous same argument, it concludes that $u^{\varepsilon}(x,t)$ is also periodic in x.

Therefore,

$$\begin{aligned} 0 &= \int_0^p \partial_x u^\varepsilon(x,t)dx \\ &= \int_{\{x | x \,\epsilon [0,p],\, \partial_x u^\varepsilon \geq 0\}} \partial_x u^\varepsilon(x,t)dx + \int_{\{x | x \,\epsilon [0,p],\, \partial_x u^\varepsilon \leq 0\}} \partial_x u^\varepsilon(x,t)dx \\ &= ITV_{[0,p]} u^\varepsilon - DTV_{[0,p]} u^\varepsilon, \end{aligned}$$

which yields

$$\begin{aligned} ITV_{[0,p]} u^\varepsilon &= DTV_{[0,p]} u^\varepsilon, \\ TV_{[0,p]} u^\varepsilon &= ITV_{[0,p]} u^\varepsilon + DTV_{[0,p]} u^\varepsilon = 2\, ITV_{[0,p]} u^\varepsilon \leq \frac{2p}{t}. \end{aligned}$$

By integration mean value theorem, one has, for any $x\epsilon[0, p]$,

$$\begin{aligned}|u^\varepsilon(x,t) - \frac{1}{p}\int_0^p u^\varepsilon(x,t)dx| &= |u^\varepsilon(x,t) - u^\varepsilon(\xi,t)| \\ \leq \quad TV_{[0,p]}u^\varepsilon \leq \frac{2p}{t},\end{aligned}$$

for some $\xi\,\epsilon[0, p]$.

From the equation (1.24), one has

$$\partial_t \int_0^p u^\varepsilon(x,t)dx = 0,$$

that is

$$\int_0^p u^\varepsilon(x,t)dx = \int_0^p u_0^\varepsilon(x)dx.$$

Noticing that

$$\frac{1}{p}\int_0^p u^\varepsilon(x,t)dx = \frac{1}{p}\int_0^p u_0^\varepsilon(x)dx = \bar{u},$$

by our regularization, we obtain

$$|u^\varepsilon(x,t) - \bar{u}| \leq \frac{2p}{t}, \quad x\,\epsilon[0,p].$$

Taking limit $\varepsilon \to 0^+$ yields the proof of the theorem.

Remark: Theorem 1.5 shows that the limit of $u(x, t)$ as $t \to 0^+$ does not depend on the initial data's oscillation and singularity. It just depends on the mean value of initial data on the periodic interval $[0, p]$.

Remark: This is very important for the homogenization theory!

§1.13 L^∞ - behavior of solution with data in $L^1 \cap L^\infty$

Theorem 1.6

Let $u(x,t)$ be the entropy weak solution of the Burgers equation with initial data $u(x,t=0) = u_0(x) \,\epsilon\, L^1 \cap L^\infty(R^1)$. Then

$$|u(x,t)|_{L^\infty(R^1)} \leq \frac{C}{\sqrt{t}}, \tag{1.26}$$

where $C = \sqrt{4m_0}$ and $m_0 = \int_{-\infty}^{\infty} |u_0(x)| dx$.

Proof: Let $u(x, t)$ be the unique entropy weak solution of the Burgers equation with initial data $u_0(x)$. Then $u(x, t) \in BV$ for $t > 0$. And due to results in $[Dp], u(x, t)$ can be regarded as a piecewise smooth function. Furthermore, by Theorem 1.4, $u(x, t) \in L^1(R^1)$ uniformly on t. For any $(x, t), t > 0$, let

$$u^{\pm}(x, t) = \lim_{\varepsilon \to 0^+} u(x \pm \varepsilon, t).$$

Now we give a pointwise estimate of $u(x,t)$ at (x,t). Define (see Figure 1.11)

$$
\begin{aligned}
y &= L^{\pm}(\tau) : y - x = u^{\pm}(x,t)(\tau - t) \\
\Omega_{+} &= \{(y,\tau) | \, 0 \le \tau \le t, \quad y \ge L^{+}(\tau)\} \\
\Omega_{-} &= \{(y,\tau) | \, 0 \le \tau \le t, \quad y \le L^{-}(\tau)\}.
\end{aligned}
$$

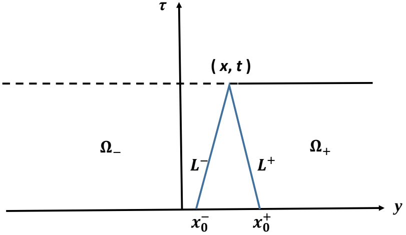

Figure 1.11

Since $u = u(y, \tau)$ is the entropy weak solution of the Burgers equation and $u(y, \tau)$ is a piecewise smooth function, one can apply divergence theorem on Ω_- to get

$$
\begin{aligned}
0 &= \oint_{\partial \Omega_-} \left(-u \, dy + \frac{u^2}{2} \, d\tau \right) \\
&= \int_{-\infty}^{x_0^-} -u_0(y) \, dy + \int_{-\infty}^{x} u(y, t) \, dy \\
&\quad + \int_0^t \left[-u \left(L^-(\tau), \tau \right) \, u^-(x, t) + \frac{(u(L^-(\tau), \tau))^2}{2} \right] d\tau,
\end{aligned}
$$

where one has used the fact that $u(-\infty, t) = 0$ for $t > 0$.

Therefore,

$$\begin{aligned}
& \left| \int_0^t \left[-u(L^-(\tau),\tau)u^-(x,t) + \frac{1}{2}(u(L^-(\tau),\tau))^2 \right] \, d\tau \right| \\
\leq & \left| \int_{-\infty}^{x_0^-} u_0(y) \, dy \right| + \left| \int_{-\infty}^{x} u(y,t) \, dy \right| \\
\leq & \int_{-\infty}^{+\infty} |u_0(y)| dy + \int_{-\infty}^{+\infty} |u(y,t)| dy \\
\leq & \; 2 \int_{-\infty}^{+\infty} |u_0(y)| dy = 2m_0.
\end{aligned} \tag{1.27}$$

On the other hand, the characteristic curve starting from the point (x, t) in backwards time for Burgers equation is the straight line with slope $u^-(x, t)$, and the entropy condition shows that this characteristic curve can be extended down to $t = 0$. So this characteristic curve is just $L^-(\tau)$. Along $L^-(\tau)$, $u(y, \tau) = u(x, t)$ for $\tau > 0$, see Figure 1.12. Thus (1.27) gives

$$\frac{1}{2}(u^-(x, t))^2 \, t \leq 2m_0,$$

i.e.

$$|u^-(x, t)| \leq \frac{2\sqrt{m_0}}{\sqrt{t}}, \quad t > 0$$

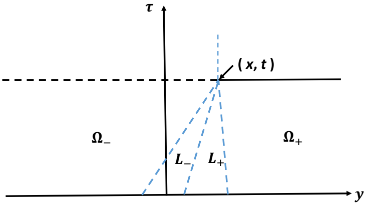

Figure 1.12

Same argument holds for $u^+(x,t)$, i.e.

$$|u^+(x,t)| \le \frac{2\sqrt{m_0}}{\sqrt{t}}, \quad t > 0.$$

It concludes that

$$||u(x,t)||_{L^\infty(R^1)} \le \frac{2\sqrt{m_0}}{\sqrt{t}}, \quad t > 0.$$

The proof of the theorem is finished.

§1.14 L_1 - behavior of solution

In this section, we are concerned with L_1 - behavior of the Burgers equation (1.4) with initial data $u_0(x) \epsilon L_c^\infty(R^1)$, i.e. $u_0(x) \epsilon L^\infty(R^1)$ and $u_0(x)$ has compact support. To conservation laws, L_1 - norm is usually more essential than other L_p - norms ($p > 1$), this is partially because of the fact that L_1 - norm of the solution provide some information on the distribution of the mass. Our goal here is to find a profile which depends only on initial data, say $U(x)$, such that

$$\int_{-\infty}^{\infty} |u(x,t) - U(x)| dx \to 0$$

as $t \to +\infty$.

Note that $U(x)$ is needed, since in general case, one cannot have

$$\int_{-\infty}^{\infty} |u(x,t)|dx \to 0 \quad , \quad t \to +\infty.$$

For example, if $u_0 > 0$, then $u(x,t) \geq 0$ and by the equation (1.4)

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_0(x)dx = m_0 \neq 0.$$

The successful construction of $U(x)$ is due to Friedrich for the Burgers equation, which is called N - wave nowadays. Now we give the construction by the following steps.

Step 1: Time-invariants of the Burgers equation
Clearly,

$$\int u(x,t)dx = \int u_0(x)dx = m_0$$

is a time - invariant quantity, which is a mass conservation.
Set

$$\begin{aligned} p(t) &= \min_x \int_{-\infty}^{x} u(y,t)dy, \\ q(t) &= \max_x \int_{x}^{+\infty} u(y,t)dy. \end{aligned}$$

We claim that both $p(t)$ and $q(t)$ are time - invariant. Since $u \,\epsilon\, Lip\,(0,\infty; L^1_{loc}(R^1))$, $p(t)$ and $q(t)$ are well-defined.

Furthermore, $u(x,t)$ is always of compact support due to the finite speed propagation (see Theorem 1.4). And since

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{x} u(y,t)dy + \int_{x}^{+\infty} u(y,t)dy,$$

we have

$$\min \int_{-\infty}^{x} u(y,t)dy = m_0 - \max \int_{x}^{+\infty} u(y,t)dy.$$

So

$$p(t) + q(t) = m_0.$$

What's more, since $u(x,t) \epsilon \, Lip \, \left(0, \infty; L^1_{loc}(R^1)\right)$, there exists a Lipschitz continuous function $x_p(t)$ such that

$$\int_{-\infty}^{x_p(t)} u(y,t)dy = p(t), \quad \int_{x_p(t)}^{+\infty} u(y,t)dy = q(t).$$

Due to entropy condition, we now prove the fact that $u(x,t)$ must be continuous at $x = x_p(t)$ and $u(x_p(t)\pm, t) = 0$.

If not, $u(x_p(t)+, t) < u(x_p(t)-, t)$ by entropy condition. So one of them must be non-zero, say $u(x_p(t)-, t)$.

If $u(x_p(t)-, t) < 0$, then $u(x_p(t)+, t) < 0$. Then there exists $\delta > 0$ such that

$$u(x,t) < 0 \quad , \quad x \, \epsilon (x_p(t), x_p(t) + \delta).$$

Consequently,

$$\begin{aligned}\int_{-\infty}^{x_p(t)+\delta} u(y,t)dy &= \int_{-\infty}^{x_p(t)} u(y,t)dy + \int_{x_p(t)}^{x_p(t)+\delta} u(y,t)dy \\ &< \int_{-\infty}^{x_p(t)} u(y,t)dy = p(t)\end{aligned}$$

This is a contradiction.

Next, if $u(x_p(t)-,t) > 0$, then there exists $\delta > 0$ such that $u(x,t) > 0 \quad , \quad x \, \epsilon \, (x_p(t)-\delta, x_p(t))$.

So

$$\begin{aligned}\int_{x_p(t)-\delta}^{+\infty} u(y,t)dy &= \int_{x_p(t)-\delta}^{x_p(t)} u(y,t)dy + \int_{x_p(t)}^{+\infty} u(y,t)dy \\ &> \int_{x_p(t)}^{+\infty} u(y,t)dy = q(t).\end{aligned}$$

This is also a contradiction. So we must have

$$u(x_p(t)-, t) = 0 = u(x_p(t)+, t).$$

Furthermore, since $u(x,t) \epsilon\, BV$ for $t > 0$, by BV function's structure (see [DR]) and by a similar argument to Theorem 1.6, it follows from Green's theorem for BV functions:

$$\begin{aligned}
\frac{d}{dt}\, p(t) &= \frac{d}{dt} \int_{-\infty}^{x_p(t)} u(y,t)dy \\
&= \int_{-\infty}^{x_p(t)} \partial_t\, u(y,t)dy + u(x_p(t),t)\, \frac{d\, x_p(t)}{dt} \\
&= -\int_{-\infty}^{x_p(t)} \partial_y \left(\frac{u^2}{2}\right) dy + u(x_p(t),t)\frac{d\, x_p(t)}{dt} \\
&= -\frac{u^2(x_p(t),t)}{2} + u(x_p(t),t)\frac{d\, x_p(t)}{dt} = 0
\end{aligned}$$

Here one has used the fact that $x_p(t) \epsilon\, Lip$ and Green theorem for BV function as mentioned previously. Therefore,

$$p(t) = p(0) = \min_x \int_{-\infty}^{x} u_0(x)dx.$$

Similarly, it yields

$$q(t) = q(0) = \max_x \int_{x}^{+\infty} u_0(x)dx.$$

Also, it is clear that $x_p(t)$ can be chosen to be $x = a$ such that

$$\begin{aligned} p(0) &= \int_{-\infty}^{a} u_0(x)dx = \min_x \int_{-\infty}^{x} u_0(y)dy, \\ q(0) &= \int_{a}^{+\infty} u_0(x)dx = \max_x \int_{x}^{+\infty} u_0(y)dy. \end{aligned}$$

Step 2: Interior Estimate

Suppose that $supp\, u_0(x) = [-N, N] \subset R^1$ without loss of generality. Otherwise, if $supp\, u_0(x) = [-N_1, N_2]$ and $N_1 \neq N_2$, one can deal with it by a simple variable transformation. Set

$$X_l(t) = \sup\{x : u(y,t) = 0, \quad \forall\, y < x\} \tag{1.28}$$
$$X_r(t) = \inf\{x : u(y,t) = 0, \quad \forall\, y > x\} \tag{1.29}$$

(See Figure 1.13)

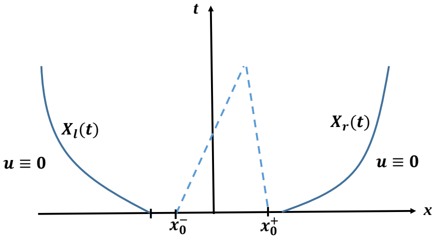

Figure 1.13

From any given interior point (x, t), one can draw the backward characteristics. The backward characteristics are straight lines due to entropy condition, which can be written as

$$\frac{x - x_0^-}{t - 0} = u(x-, t), \quad \frac{x - x_0^+}{t - 0} = u(x+, t).$$

Thus,

$$u(x\pm, t) = \frac{x}{t} - \frac{x_0^\pm}{t}, \quad X_l(t) < x < X_r(t), \quad -N \le x_0^* \le N. \tag{1.30}$$

Step 3: Upper estimate of the support

Now we analyze how the support of $u(x,t)$ develops. Take a point $(X_r(t),t)$ for example, draw the left backward characteristic from $(X_r(t),t)$ which is a straight line. Let Ω be the region depicted in Figure 1.14, defined by the backward characteristic, the line $\tau = t$ and the x-axis. From the definition of $X_r(t)$, $u = 0$ on the right of $X_r(t)$ in Figure 1.14, and $X_r(t)$ satisfies

$$\begin{cases} \frac{d\,X_r(t)}{dt} = \frac{1}{2}(u(X_r(t)-,t)+0), \\ X_r(0) = N. \end{cases} \tag{1.31}$$

Noticing that the left backward characteristic starting from $(X_r(t),t)$ is

$$\frac{y - X_r(t)}{\tau - t} = u(X_r(t)-,t),$$

i.e.

$$\frac{dy}{d\tau} = u(X_r(t)-, t), \quad y(t) = X_r(t) - .$$

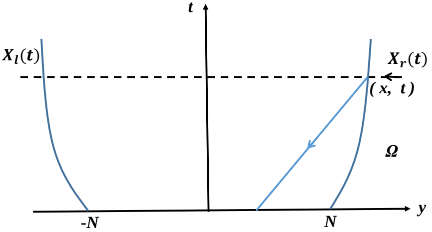

Figure 1.14

Here we use (y, τ) replacing of (x, t) to make it clear. Then because of the reason mentioned in Theorem 1.6, one can apply the Green theorem to the solution $u(x, t)$ of the Burgers equation in Ω to get

$$
\begin{aligned}
0 &= \iint_{\Omega}\left[\partial_\tau u + \partial_y\left(\frac{u^2}{2}\right)\right] dy\, d\tau \\
&= \int_{\partial\Omega}(-u)dy + \frac{u^2}{2}d\tau \\
&= \int_0^t \left(u\frac{dy}{d\tau} - \frac{u^2}{2}\right) d\tau - \int_{y_0}^{+\infty} u_0(y)dy \\
&= \frac{1}{2}u^2(X_r(t)-, t)t - \int_{y_0}^{+\infty} u_0(y)dy,
\end{aligned}
$$

which holds true due to the fact that along the characteristic $y = y(\tau)$, $u(y(\tau), \tau) = u(X_r(t)-, t)$. Therefore,

$$
\begin{aligned}
\frac{1}{2}u^2(X_r(t)-,t)t &= \int_{y_0}^{+\infty} u_0(y)dy \le q, \\
|u(X_r(t)-,t)| &\le \sqrt{\frac{2q}{t}}.
\end{aligned}
$$

This, together with (1.31), implies

$$
X_r(t) \le N + \sqrt{2qt}. \tag{1.32}
$$

Similarly, one can get

$$
X_l(t) \ge -N - \sqrt{-2pt}. \tag{1.33}
$$

Step 4: Lower estimate of the support

One can also obtain the following lower estimate on the support.

$$X_r(t) \geq -N + \sqrt{2qt}, \tag{1.34}$$

$$X_l(t) \leq N - \sqrt{-2pt}. \tag{1.35}$$

Indeed, from the definition of $q(t)$, there exists a number $a \,\epsilon [-N, N]$ such that

$$q(t) = \int_a^{+\infty} u(x,t)dx = \int_a^{X_r(t)} u(x,t)dx = \int_a^{X_r(t)} \left(\frac{x}{t} - \frac{x_0}{t}\right) dx,$$

where the interior estimate (1.30) has been used and $x_0 \epsilon [-N, N]$.
It follows from the property of a and x_0 that
$q(t) \leq \int_a^{X_r(t)} \frac{(x-a)}{t} dx \leq \frac{1}{2t}(X_r(t) - a)^2$.

Thus

$$X_r(t) \geq a + \sqrt{2qt} \geq -N + \sqrt{2qt}$$

(1.34) is then proved. Similarly, (1.35) can be obtained.

Step 5: N-wave

Let $X_l(t), X_r(t)$ be defined as before. Define

$$N(x,t,N,p,q) = \begin{cases} \frac{x}{t}, & X_l(0) - \sqrt{-2pt} < x < X_r(0) + \sqrt{2qt} \\ 0, & \text{otherwise.} \end{cases} \tag{1.36}$$

According to our assumption, $X_l(0) = -N, X_r(0) = N$ here. The function $N(x,t,N,p,q)$ is called an "N-wave", because of its profile at each fixed $t > 0$ (See Figure 1.15). Clearly,

$$\begin{array}{ll} \int_{-\infty}^{0} N(x,t,N,p,q)dx = p + O(\frac{1}{\sqrt{t}}), & t > 0 \\ \int_{0}^{+\infty} N(x,t,N,p,q)dx = q + O(\frac{1}{\sqrt{t}}), & t > 0 \end{array}$$

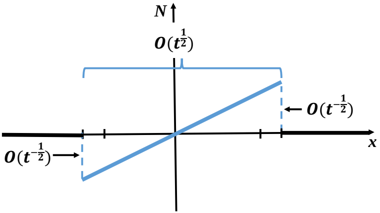

Figure 1.15

Now we state the L^1-asymptotic of entropy weak solution $u(x, t)$ for the Burgers equation.

Theorem 1.7

$$\int_{-\infty}^{+\infty} |u(x,t) - N(x,t,N,p,q)|dx = \frac{O(1)}{\sqrt{t}}. \tag{1.37}$$

Proof:

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} |u(x,t) - N(x,t,N,p,q)|dx \\
= &\left\{ \int_{-\infty}^{X_l(0)-\sqrt{-2pt}} + \int_{X_l(0)-\sqrt{-2pt}}^{X_l(t)} + \int_{X_l(t)}^{X_r(t)} + \int_{X_r(t)}^{X_r(0)+\sqrt{2qt}} \right. \\
&\left. + \int_{X_l(0)+\sqrt{2qt}}^{+\infty} \right\} |u(x,t) - N(x,t,N,p,q)|dx \\
\equiv &\; I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

By estimates (1.32) and (1.33), and the definition of N-wave, it is clear that

$$
\begin{aligned}
I_1 &= I_5 = 0 \\
I_2 &= \int_{X_l(0)-\sqrt{-2pt}}^{X_l(t)} \left| u(x,t) - \frac{x}{t} \right| dx
\end{aligned}
$$

Due to L^∞- behavior of the solution $u(x,t)$ stated in Theorem 1.6, one has

$$
|u(x,t)|_{L^\infty} \leq \frac{C}{\sqrt{t}}.
$$

It follows from the upper estimate (1.33) and lower estimate (1.35) on the support that

$$
|X_l(t) - (X_l(0) - \sqrt{-2pt})| \leq 2N,
$$

where $-N = X_l(0)$. And when t is large enough, both $X_l(t)$ and $X_l(0) - \sqrt{-2pt}$ are negative. So when t is large enough, one has

$$
\begin{aligned}
I_2 &\leq O(1) \cdot \frac{1}{\sqrt{t}} + \left| \int_{X_l(t)}^{X_l(0)-\sqrt{-2pt}} \frac{x}{t} dx \right| \\
&= O(1) \cdot \frac{1}{\sqrt{t}} + \frac{1}{2t} \left| \left(X_l(0) - \sqrt{-2pt} \right)^2 - X_l^2(t) \right| \\
&\leq O(1) \cdot \frac{1}{\sqrt{t}} + \frac{1}{2t} 2N \left| \left(X_l(0) - \sqrt{-2pt} \right) + X_l(t) \right| \\
&\leq \frac{O(1)}{\sqrt{t}}.
\end{aligned}
$$

Similarly, one has

$$I_4 \leq \frac{O(1)}{\sqrt{t}}.$$

Furthermore, according to the interior estimate (1.30), one yields

$$I_3 \leq \left|\frac{x_0^{\pm}}{t}\right| \cdot |X_r(t) - X_l(t)| \leq O(1) \cdot \frac{1}{\sqrt{t}}.$$

Therefore (1.37) is proved and the proof of the theorem is finished.

Remark: Can one prove Theorem 1.2 by vanishing viscosity method as in §1.12?

§1.15 Asymptotics toward shocks and rarefaction waves

Now we consider long time behavior for the following problem (shock wave case).

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0, \\ u(x, t = 0) = u_0(x), \\ \lim_{x \to \pm\infty} u_0(x) = u_\pm, \quad u_- > u_+. \end{cases} \tag{1.38}$$

Physically, there is a shock wave at infinity. Then what is the large asymptotic of the entropy weak solution for the Burgers equation? Set

$$u_s = \begin{cases} u_-, & x < st, \\ u_+, & x > st, \end{cases}$$

where $s = \frac{1}{2}(u_+ + u_-)$.

Then we claim that $u(x,t)$ tends to a shifted shock of u_s, denoted by $U_s = u_s(x - x_0, t)$. The shift x_0 is defined by

$$x_0 = \frac{m_0}{u_- - u_+},$$

where

$$m_0 = \int (u_0(x) - u_s(x, t = 0))dx. \tag{1.39}$$

Actually, it follows from the Burgers equation and the profile of shifted shock $U_s(x,t)$ that

$$\begin{aligned}
& \int (u(x,t) - U_s(x,t))\, dx = \int (u_0(x) - U_s(x, t=0))\, dx \\
= & \int (u_0 - u_s(x, t=0))\, dx + \int (u_s(x, t=0) - U_s(x, t=0))\, dx \\
= & \; m_0 + (u_+ - u_-)x_0 = 0.
\end{aligned}$$

So if $u(x,t) \to U_s(x,t)$ as $t \to +\infty$, x_0 should be equal to $\frac{m_0}{u_- - u_+}$, which is (1.39), see Figure 1.16.

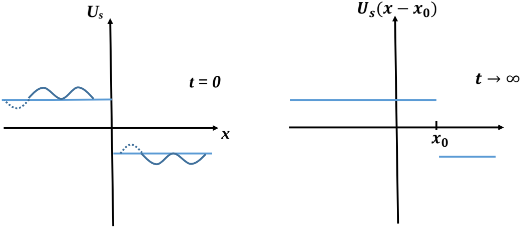

Figure 1.16

In order to make the idea clearer, we consider the following simpler case: there exists an $N > 0$, such that

$$u_0(x) = \begin{cases} u_-, & x < -N, \\ u_+, & x > N. \end{cases}$$

Then we have

Theorem 1.8
Let U_s be the shifted shock determined by the initial excessive mass. Then there exists a $T_* > 0$ such that

$$u(x, t) = U_s(x, t) \quad \text{for} \quad t > T_*.$$

Proof: Let $X_l(t)$ be the minimal characteristic from $(-N, 0)$ and $X_r(t)$ be the maximal characteristic from $(N, 0)$ (see Figure. 1.17).

It suffices to show that there exists $T_* > 0$ such that

$$X_l(T_*) = X_r(T_*).$$

Set

$$D(t) = X_r(t) - X_l(t).$$

Then

$$\begin{aligned}\frac{d}{dt}D(t) &= \dot{X}_r(t) - \dot{X}_l(t)\\ &= \frac{1}{2}(u_+ + u(X_r(t)-,t)) - \frac{1}{2}(u_- + u(X_l(t)+,t))\\ &= \frac{1}{2}(u_+ - u_-) + \frac{1}{2}(u(X_r(t)-,t) - u(X_l(t)+,t)),\end{aligned}$$

Draw the backward characteristic from $(X_l(t), t)$ and $(X_r(t), t)$ respectively (see Figure 1.17). Then one has

$$\begin{aligned}u(X_r(t)-,t) &= \frac{X_r(t)}{t} - \frac{x_0^r}{t},\\ u(X_l(t)+,t) &= \frac{X_l(t)}{t} - \frac{x_0^l}{t}.\end{aligned}$$

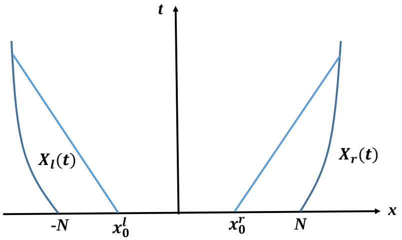

Figure 1.17

So

$$u(X_r(t)-,t) - u(X_l(t)+,t) = \frac{1}{t}D(t) - \frac{x_0^r - x_0^l}{t},$$

and

$$\frac{d}{dt}D(t) = \frac{1}{2}(u_+ - u_-) + \frac{1}{2t}D(t) - \frac{x_0^r - x_0^l}{2t}.$$

Since $x_0^r > x_0^l$, one has

$$\frac{d}{dt}D(t) \le \frac{1}{2}(u_+ - u_-) + \frac{1}{2t}D(t).$$

That is

$$\frac{d}{dt}\left(\frac{D(t)}{\sqrt{t}}\right) \le \frac{1}{2\sqrt{t}}(u_+ - u_-).$$

Integrating on t from 1 to t yields

$$D(t) \le D(1)\sqrt{t} + (u_+ - u_-)(t - \sqrt{t}).$$

Since $u_- > u_+$, it follows from the above inequality that there exists $T_* > 0$ such that $D(T^*) = 0$.

The proof of the theorem is finished.

From the above theorem, one can see that shock waves absorbs all the other disturbances in a Riemann solution, no matter how the perturbation of the initial data in the compact set. It shows that shock is strongly stable.

Another case is $u_+ > u_-$. Consider the Burgers equation

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0 \\ u(x, t = 0) = u_0(x) \\ \lim_{x \to \pm\infty} u_0(x) = u_\pm \end{cases} \tag{1.40}$$

and the centered rarefaction wave

$$u_R(x,t) = \begin{cases} u_- & \text{if} \quad x < u_- \, t \\ \frac{x}{t} & \text{if} \quad u_- \, t < x < u_+ \, t \\ u_+ & \text{if} \quad x > u_+ \, t \end{cases}$$

satisfying

$$\begin{cases} \partial_t \, u + \partial_x \left(\frac{u^2}{2} \right) = 0 \\ u(x, t=0) = \begin{cases} u_- & \text{if} \quad x < 0 \\ u_+ & \text{if} \quad x > 0 \end{cases} \\ u_+ > u_- \end{cases}$$

we can prove that the general solution of (1.40) with $u_+ > u_-$ will eventually tends to the centered rarefaction wave with some rate of decay in time.

Theorem 1.9
Assume that $u_0(x) = u_-$ for $x < -N$; $\quad u_0(x) = u_+$ for $x > N$; $\quad u_- < u_+$. Then the unique solution u to (1.40) satisfies

$$|u(x,t) - u_R(x,t)| \leq \frac{C}{\sqrt{t}}$$

where C depends only on N.

Remark: If $N = +\infty$, one would not expect the uniform decay. However, one can prove that $|u - u_R| \to 0$ as $t \to \infty$ if $u_0(x) \to u_R(x)$ as $x \to \infty$ sufficiently fast in x. For example, $|u_0(x) - u_R(x)| \leq C\, e^{-\alpha x}$ for some $\alpha > 0$.

Proof of Theorem 1.9: Let
$X_l(t) = \sup\{x : u(y,t) = u_- \quad \forall\, y < x\}$ and
$X_r(t) = \inf\{x : u(y,t) = u_+ \quad \forall\, y > x\}$ be two sided characteristics starting from $X_l(0) = -N$ and $X_r(0) = N$, respectively, see Figure 1.17. First we estimate the propagation of $X_l(t)$ and $X_r(t)$ in t.

Step 1: Estimate of the essential support

Let $D(t) = X_r(t) - X_l(t)$ be the length of the interior we are interesting, and set $L(t) = D(t) - (u_+ - u_-)t$. The goal is to prove $|L(t)| \leq O(1)\sqrt{t}$ for $t > 1$, where the constant depends only on N. To get this estimate, one computes

$$\frac{d}{dt} L(t) = \dot{D}(t) - (u_+ - u_-) = \dot{X}_r(t) - \dot{X}_l(t) - (u_+ - u_-)$$

where the notation $\dot{} = \frac{d}{dt}$. To find $\dot{X}_l(t), \dot{X}_r(t)$, we draw a backward characteristic from $(X_l(t), t)$ to some point $(X_l^o, 0)$, and a backward characteristic from $(X_r(t), t)$ to $(X_r^o, 0)$. Then

$$\begin{aligned} \dot{X}_r(t) &= \frac{1}{2}\left(u(X_r(t)-, t) + u(X_r(t)+, t)\right) \\ &= \frac{1}{2}\left(u_+ + \frac{X_r(t) - X_r^o}{t}\right) \end{aligned}$$

in either u is continuous or a shock at $x = X_r(t)$. Note that $u(X_r(t)-, t) = \frac{X_r(t) - X_r^o}{t}$ since the backward characteristic does not intersect any shocks by entropy condition. Similarly,

$$\begin{aligned}\dot{X}_l(t) &= \frac{1}{2}(u(X_l(t)-,t)+u(X_l(t)+,t)) \\ &= \frac{1}{2}\left(u_- + \frac{X_l(t)}{t} - \frac{X_l^o}{t}\right)\end{aligned}$$

Hence,

$$\begin{aligned}\frac{d}{dt}L(t) &= -\frac{1}{2}(u_+ - u_-) + \frac{1}{2t}(X_r(t) - X_l(t)) - \frac{1}{2t}(X_r^o - X_l^o) \\ &= \frac{1}{2t}L(t) - \frac{1}{2t}(X_r^o - X_l^o)\end{aligned}$$

Multiply $\frac{1}{\sqrt{t}}$ and integrate in t to get

$$\begin{aligned}|L(t)| &\leq L(1)\sqrt{t} + (1-\sqrt{t})(X_r^o - X_l^o) \\ &\leq C_1(N)\sqrt{t}\end{aligned}$$

Unfortunately, it is not enough to estimate $X_l(t)$ and $X_r(t)$. We need the following step.

Step 2: Estimate of total decreasing variation

Let any partition on $(X_l(t), X_r(t))$ at time t,
$X_l(t) = X_0(t) < X_1(t) < X_2(t) < \cdots < X_n(t) = X_r(t)$. Since the increasing total variation is defined by

$$IV\ u(\cdot, t) = \sup_{P:partition,\ P=\{x_0<x_1<x_2<\cdots<x_n\}} \sum_i \max\{u(x_{i+1}, t) - u(x_i, t), 0\}.$$

Let $X_{i+1}(t')$ be the backward characteristic from $X_{i+1}(t)$ which intersects with $\tau = 0$ at X^0_{i+1}. See Figure 1.18. Then one can find that

$$
\begin{aligned}
& \sum_{i:\, u(X_{i+1}(t)-,t)>u(X_i(t)+,t)} u(X_{i+1}(t)-,t) - u(X_i(t)+,t) \\
= & \sum_{i:\, u(X_{i+1}(t)-,t)>u(X_i(t)+,t)} \frac{X_{i+1}(t) - X_{i+1}^o}{t} - \frac{X_i(t) - X_i^o}{t} \\
= & \sum_{i:\, u(X_{i+1}(t)-,t)>u(X_i(t)+,t)} \frac{X_{i+1}(t) - X_i(t)}{t} - \frac{X_{i+1}^o - X_i^o}{t} \\
\leq & \sum_i \frac{X_{i+1}(t) - X_i(t)}{t} = \frac{D(t)}{t}
\end{aligned}
$$

since $X_i^o < X_{i+1}^o$ for any $i = 0, 1, \cdots, n-1$ by the fact $X_i(t^{'})$ will not intersect $X_{i+1}(t^{'})$ for $0 \leq t^{'} \leq t$. Hence $IV\, u(\cdot, t) \leq \frac{D(t)}{t}$.

Now $u_+ - u_- = IV - DV$, where the decreasing total variation is

$$DV\, u(\cdot, t) = - \sup_{P:partition} \sum_i \min\{u(x_{i+1}, t) - u(x_i, t), 0\}$$

We deduce that
$DV = IV - (u_+ - u_-) \leq \frac{L(t)+(u_+-u_-)t}{t} - (u_+ - u_-) \leq \frac{O(1)}{\sqrt{t}}$

Note that one cannot get this decay in time of total decreasing variation directly from the entropy condition. Here the estimating decreasing variation from jumping down across discontinuity by estimating the expanding of rarefaction waves.

So

$$\begin{aligned} u_- \geq \dot{X}_l(t) &= u_- + \frac{1}{2}(u(X_l(t)+, t) - u_-) \\ &\geq u_- - \frac{1}{2} DV\, u(\cdot, t) \\ &\geq u_- - \frac{O(1)}{\sqrt{t}} \end{aligned}$$

and then integrate over time from 0 to t, we get

$$X_l(t) = -N + u_-\, t + O(1)\sqrt{t}$$

similarly we have $X_r(t) = N + u_+\, t + O(1)\sqrt{t}$.

Now we are ready to prove for the convergence of the solutions to the centered rarefaction wave.

Step 3: Convergence

We start at a large time t. Without loss of generality, we may assume $u_-\, t < X_l(t) < u_+\, t < X_r(t)$ at time t. Since $u(x,t)$ and $u_R(x,t)$ have same constant states in $x < u_-\, t$ and $x > X_r(t)$, we consider the following regions $\mathrm{I} = \{x : u_-\, t < x < X_l(t)\}$, $\mathrm{I}I = \{x : X_l(t) < x < u_+\, t\}$, $\mathrm{I}II = \{x : u_+\, t < x < X_r(t)\}$.

Case 1: $x \,\epsilon \mathrm{I}$. Then x lies in the rarefaction wave and $u_R(x,t) = \frac{x}{t}$, also $x < X_l(t), \quad u(x,t) = u_-$. Hence

$$|u(x,t) - u_R(x,t)| = \left| u_- - \frac{x}{t} \right| = \frac{x - u_-\, t}{t} < \frac{X_l(t) - u_- t}{t} \leq \frac{O(1)}{\sqrt{t}}$$

Case 2: $x \,\epsilon \mathrm{I}I$. Then $u_R(x,t) = \frac{x}{t}$, and by drawing the backward characteristic to $(x_0, 0)$, $u(x,t)$ is constant on this line and the slope of this line is $u(x,t) = \frac{x}{t} - \frac{x_0}{t}$. Therefore

$$|u(x,t) - u_R(x,t)| = \frac{|x_0|}{t} = \frac{O(1)}{\sqrt{t}}$$

Case 3: $x \epsilon \mathrm{I}II$. x lies outside the rarefaction wave section and $u_R(x,t) = u_+$. Similar to case 2 we have $u(x,t) = \frac{x-x_0}{t}$. Then

$$\begin{aligned} |u(x,t) - u_R(x,t)| &= \left| \frac{x - x_0}{t} - u_+ \right| \\ &\leq \frac{x - u_+ \, t}{t} + \frac{|x_0|}{t} \\ &\leq \frac{X_r(t) - u_+ \, t}{t} + \frac{O(1)}{t} \\ &= \frac{N + O(1)\sqrt{t}}{t} + \frac{O(1)}{t} \\ &\leq \frac{O(1)}{\sqrt{t}} \end{aligned}$$

It follows from the estimates of all cases that $|u(x,t) - u_R(x,t)| \leq \frac{C(N)}{\sqrt{t}}$. The theorem is proved.

$$
\begin{aligned}
u(X_r(t)-, t) &= \frac{X_r(t)}{t} - \frac{x_0^r}{t}, \\
u(X_l(t)+, t) &= \frac{X_l(t)}{t} - \frac{x_0^l}{t}.
\end{aligned}
$$

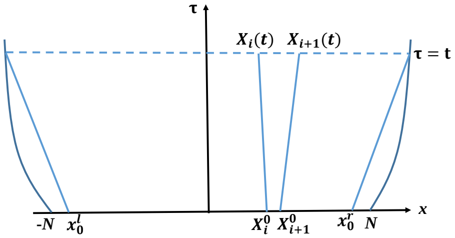

Figure 1.18