Math 2230B, Complex Variables with Applications

1. Let C denote the positively oriented boundary of the square whose slides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

$$\text{(a) } \int_C \frac{e^{-z}dz}{z-(\pi i/2)}; \quad \text{(b) } \int_C \frac{\cos z}{z(z^2+8)}dz; \quad \text{(c) } \int_C \frac{zdz}{2z+1}$$
$$\text{(d) } \int_C \frac{\cosh z}{z^4}dz; \quad \text{(e) } \int_C \frac{\tan(z/2)}{(z-x_0)^2}dz \quad (-2 < x_0 < 2)$$

2. Find the value of the integral of g(z) around the circle $|z - i| = 2$ in the positive sense when

$$\text{(a) } g(z) = \frac{1}{z^2+4}; \quad \text{(b) } g(z) = \frac{1}{(z^2+4)^2}.$$

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}ds \quad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s-z)^3}ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Let C be the unit circle $z = e^{i\theta}(-\pi \leq \theta \leq \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi.$$

6. Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.
Suggestion: Use Cauchy's inequality (Sec. 57) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

7. Suppose that f(z) is entire and that the harmonic function u(x,y)=Re[f(z)] has an upper bound u_0; that is, $u(x, y) \leq u_0$ for all points (x,y) in the xy plane. Show that u(x,y) must be constant throughout the plane.
*Suggestion:*Apply Liouville's theorem to the function $g(z) = \exp[f(z)]$.

8. Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a minimum value m in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values to the function $g(z) = \frac{1}{f(z)}$.

9. Use the function $f(z) = z$ to show that in exercise 8 the condition $f(z) \neq 0$ anywhere in R is necessary in order to obtain the result of that exercise. That is, show that $|f(z)|$ can reach its minimum value at an interior point when the minimum value is zero.

10. Let R region $0 \leq x \leq \pi$, $0 \leq y \leq 1$ (Fig. 72). Show that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R at the boundary point $z = (\pi/2) + i$.
Suggestion: Write $|f(z)|^2 = \sin^2 x + \sinh^2 y$(see Sec.37) and locate points in R at which $\sin^2 x$ and $\sinh^2 y$ are the largest.

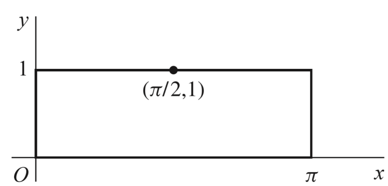

FIGURE 72

11. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function u(x,y) has a minimum value in R which occurs on the boundary of R and never in the interior.

12. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \pi$. Illustrate results in Sec. 59 and exercise 5 by finding points in R where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.