Math 2230A, Complex Variables with Applications

1. Use residues to establish the following integration formula:

   (a)
   $$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}.$$

   (b)
   $$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\pi.$$

   (c)
   $$\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5-4\cos 2\theta} = \frac{3\pi}{8}.$$

   (d)
   $$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad (-1 < a < 1).$$

2. Let C denote the unit circle $|z| = 1$, described in the positive sense. Use the theorem in Sec. 93 to determine the value of $\Delta_C \arg f(z)$ when

   (a) $f(z) = z^2$; (b) $f(z) = 1/z^2$; (c) $f(z) = (2z-1)^7/z^3$.

3. Let f be a function which is analytic inside and on a positively oriented simple closed contour C, and suppose that f(z) is never zero on C. Let the image of C under the transformation $\omega = f(z)$ be the closed contour Γ shown in Fig. 107. Determine the value of $\Delta_C \arg f(z)$ from that figure; and, with the aid of the theorem in Sec. 93, determine the number of zeros, counting multiplicities, of f interior to C.

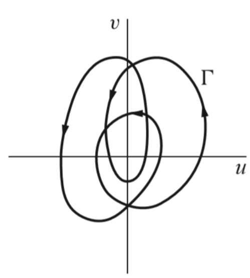

FIGURE 107

4. Using the notation in Sec. 93, suppose that Γ does not enclose the origin $\omega = 0$ and that there is a ray from that point which does not intersect Γ. By observing that the absolute value of $\Delta_C \arg f(z)$ must be less than 2π

when a point z makes one cycle around C and recalling that $\Delta_C \arg f(z)$ is an integral multiple of 2π, point out why the winding number of Γ with respect to the origin $\omega = 0$ must be zero.

5. Suppose that a function f is analytic inside and on a positively oriented simple closed contour C and that it has no zeros on C. Show that if f has n zeros $z_k \quad (k = 1, 2, ..., n)$ inside C, where each z_k is of multiplicity m_k, then
$$\int_C \frac{zf'(z)}{f(z)} dz = 2\pi i \Sigma_{k=1}^n m_k z_k.$$
[Compare with equation (8), Sec. 93, when $P = 0$ there.]

6. Determine the number of zeros, counting multiplicities, of the polynomial

(a) $z^6 - 5z^4 + z^3 - 2z$; (b) $2z^4 - 2z^3 + 2z^2 - 2z + 9$; (c) $z^7 - 4z^3 + z - 1$.

inside the circle $|z| = 1$.

7. Determine the number of roots, counting multiplicities, of the equation
$$2z^5 - 6z^2 + z + 1 = 0$$
in the annulus $1 \leqslant |z| < 2$.