Math 2230B, Complex Variables with Applications

Use residues to derive the integration formulas in Question 1 and Question 2.

1.
$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

2.
$$\int_0^\infty \frac{x^2dx}{x^6+1} = \frac{\pi}{6}$$

3. Use a residue and a contour shown in Fig.95, where $R > 1$, to establish the integration formula
$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$$

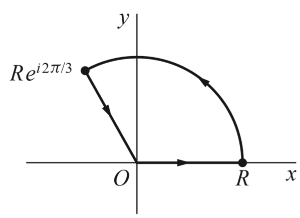

FIGURE 95

4. Use residues to derive the integration formula
$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}dx = \frac{\pi}{4b^3}(1+ab)e^{-ab} \quad (a>0, b>0).$$

5. Use residues to derive the integration formula
$$\int_{-\infty}^\infty \frac{x^3\sin ax}{x^4+4}dx = \pi e^{-a}\cos a \quad (a>0).$$

6. Use residues to find the Cauchy principal values of the improper integrals
$$\int_{-\infty}^\infty \frac{\sin x dx}{x^2+4x+5}.$$

7. Follow the steps below to evaluate the *Fresnel integrals*, which are important in diffraction theory:

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

(a) By integrating the function $\exp(iz^2)$ around the positively oriented boundary of the sector $0 \le r \le R$, $0 \le \theta \le \frac{\pi}{4}$ (Fig. 99) and appealing to the Cauchy-Goursat theorem, show that

$$\int_0^R \cos\left(x^2\right) dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2} dr - \operatorname{Re}\int_{C_R} e^{iz^2} dz$$

and

$$\int_0^R \sin\left(x^2\right) dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2} dr - \operatorname{Im}\int_{C_R} e^{iz^2} dz,$$

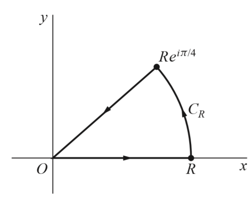

FIGURE 99

where C_R is the arc $z = Re^{i\theta}(0 \le \theta \le \frac{\pi}{4})$.

(b) Show that the value of the integral along the arc C_R in part (a) tends to zero as R tends to infinity by obtaining the inequality

$$\left|\int_{C_R} e^{iz^2} dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2 \sin\phi} d\phi$$

and then referring to the form (2), Sec. 81, of Jordan's inequality.

(c) Use the results in part (a) and (b), together with the knowing formula

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

to complete the exercise.

8. Use the function $f(z) = (e^{iaz} - e^{ibz})/z^2$ and the indented contour in Fig.108(Sec.89) to derive the integration formula

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b-a) \quad (a \ge 0, b \ge 0)$$

Then with the aid of the trigonometric identity $1 - \cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

9. Derive the integration formula

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating the function

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1} \quad (|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2})$$

over the indented contour appearing in Fig. 109 (Sec. 90).

10. The **beta function** is this function of two real variables:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt \quad (p > 0, q > 0).$$

Make the substitution $t = 1/(x+1)$ and use the result obtained in the example in Sec. 91 to show that

$$B(p, 1-p) = \frac{\pi}{\sin(p\pi)} \quad (0 < p < 1).$$