Math 2230A, Complex Variables with Applications

1. Use established properties of moduli to show that when $|z_3| \neq |z_4|$,
$$\frac{Re(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}.$$

2. Verify that $\sqrt{2}|z| \geq |Rez| + |Imz|$.

3. In each case, sketch the set of points determined by the given condition: (a) $|z-1+i| = 1$ (b) $|z+i| \leq 3$ (c) $|z-4i| \geq 4$

4. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that $|z-1| = |z+i|$ represents the line through the origin whose slope is -1.

5. Use properties of conjugates and moduli established in Sec.6 to show that
(a) $\overline{\bar{z}+3i} = z - 3i$
(b) $\overline{iz} = -i\bar{z}$
(c) $\overline{(2+i)^2} = 3 - 4i$
(d) $|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}|2z+5|$.

6. Sketch the set of points determined by the condition
(a) $\text{Re}(\bar{z}-i) = 2$
$(b)|2\bar{z}+i| = 4$.

7. Show that
$$\left|\text{Re}\left(2+\bar{z}+z^3\right)\right| \leq 4 \quad \text{when } |z| \leq 1$$

8. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (2), Sec. 5, show that if z lies on the circle $|z| = 2$, then
$$\left|\frac{1}{z^4-4z^2+3}\right| \leq \frac{1}{3}.$$

9. Find the principal argument Argz when
(a) $z = \frac{-2}{1+\sqrt{3}i}$; (b) $z = (\sqrt{3}-i)^6$.

10. Show that $(a)\left|e^{i\theta}\right| = 1$; $\quad (b)\overline{e^{i\theta}} = e^{-i\theta}$.

11. Using the fact that the modulus $|e^{i\theta}-1|$ is the distance between the points $e^{i\theta}$ and 1(see Sec. 4), give a geometric argument to find a value of θ in the interval $0 \leq \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1 = 2$.

12. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that
(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$
(b) $5i/(2+i) = 1+2i$
(c) $(\sqrt{3}+i)^6 = -64$
(d) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

13. Let z be a nonzero complex number and n a negative integer(n=-1, -2,...). Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, ...$Using the expressions

$$z^m = r^m e^{im\theta} \quad \text{and} \quad z^{-1} = \left(\frac{1}{r}\right) e^{i(-\theta)}$$

verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec.7 could have been written alternatively as $z^n = (z^m)^{-1}$

14. Establish the identity

$$(z \neq 1)$$
$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

and then use it to derive Lagrange's trigonometric identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \quad (0 < \theta < 2\pi).$$

15. Find the square root of (a) $2i; (b) 1-\sqrt{3}i$ and express them in rectangular coordinates.

16. Find the three cube roots c_k(k=0,1,2) of -8i, express them in rectabgular coordinates, and point out why they are as shown in Fig.12.

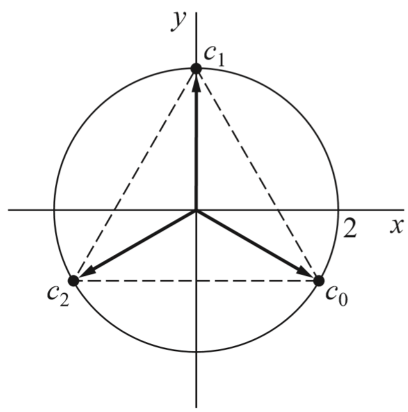

FIGURE 12

17. Find $(-8-8\sqrt{3}i)^{1/4}$, express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out which is the principal root.

18. In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root: (a) $(-1)^{1/3}$ (b) $8^{1/6}$.

19. Sketch the following sets and determine which are domains:
(*a*)$|z-2+i| \leq 1$
(b) $|2z+3| > 4$
(c) $\operatorname{Im} z > 1$
(d) $\operatorname{Im} z = 1$
(e) $0 \leq \arg z \leq \pi/4 (z \neq 0)$
(f) $|z-4| \geq |z|$.

20. In each case, sketch the closure of the set:
(*a*) $-\pi < \arg z < \pi (z \neq 0)$
(b) $|\operatorname{Re} z| < |z|$
(*c*) $\operatorname{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$
(*d*) $\operatorname{Re}(z^2) > 0$.