THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 8
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Theorem 1. *(Taylor Series) Suppose that* f *is analytic in a disk* $\{z \in \mathbb{C} \mid |z - z_0| < R\}$*. Then* f *has the power series representation centred at* $z = z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad \textit{for all } z \in \{z \in \mathbb{C} \mid |z - z_0| < R\}.$$

Remark : The Taylor series of f centred at a given point is unique. (a_n is unique)

Remark : This means that the infinite series converges for any z in the disk. (It may not be uniform! You may check by Weierstrass M-test.)

Remark : If f is analytic at some point z_0, then it must be analytic in some small disk $\{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$ such that we have a convergent Taylor series there.

Remark : If f is entire, then the Taylor series converges in the domain $\mathbb{C} = \{z \in \mathbb{C} \mid |z - z_0| < \infty\}$ for any z_0.

Suppose we have a function f which admits a singularity at $z = z_0$ such that $\lim_{z \to z_0} |f(z)| = \infty$.(Or other types of singularity at which $f(z)$ is not welled-defined, we will discuss later.) It is clear that we do not have a Taylor Series for f centred at $z = z_0$ since $a_0 = f(z_0)$ is not defined! ($a_n = \dfrac{f^{(n)}(z_0)}{n!}$ are defined as well!)

Theorem 2. *(Laurent Series) Suppose that* f *is analytic in an annulus* $\{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$*, then* f *has the power series representation centred at* $z = z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \textit{for all } z \in \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}.$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)dz}{(z - z_0)^{n+1}}$ $(n = 0, 1, ...)$ *and* $b_n = \dfrac{1}{2\pi i} \displaystyle\int_C f(z)(z - z_0)^{n-1} dz$ $(n = 1, 2, ...)$*.* C *is any closed contour in the annulus.*

Remark : The formula for a_n and b_n here may be difficult to compute.

An important technique to compute the whole Laurent series is the following proposition :

Proposition 1. *(Geometric Sum) If* $|z| < 1$*, then* $\dfrac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$*.*

Example 1. *Find the Laurent series of* $f = \dfrac{1}{z^2 + 4}$ *centred at* $z = 2i$ *in the region* $\{|z - 2i| > 4\}$

First, we observe that $\dfrac{1}{z^2 + 4} = \left(\dfrac{1}{z - 2i}\right)\left(\dfrac{1}{z + 2i}\right)$. We shall be careful that $z = 2i$ is a singularity of f in the region so it makes sense to consider the Laurent series of f. If we can find the Laurent series for $\dfrac{1}{z + 2i}$, then it is done since $\dfrac{1}{z - 2i}$ is already 'good'.

Second we find the Laurent series for $\dfrac{1}{z + 2i}$ by proposition 1. We observe that

$$\frac{1}{z + 2i} = \frac{1}{z - 2i + 4i} = \frac{1}{z - 2i} \frac{1}{\left(1 - \left(-\dfrac{4i}{z - 2i}\right)\right)}$$

Since $4 < |z-2i| \Rightarrow \left|\frac{4i}{z-2i}\right| < 1$. By proposition 1,

$$\frac{1}{\left(1-\left(-\frac{4i}{z-2i}\right)\right)} = \sum_{n=0}^{\infty}\left(-\frac{4i}{z-2i}\right)^n$$

Therefore,

$$f = \frac{1}{z^2+4} = \left(\frac{1}{z-2i}\right)\left(\frac{1}{z+2i}\right) = \sum_{n=0}^{\infty}\frac{(-4i)^n}{(z-2i)^{n+2}}$$

Example 2. *Try to find a Laurent series of example 1 in the region* $\{0 < |z-2i| < 4\}$.

Example 3. *Find the Laurent series of* $\frac{1}{z\sin z}$ *in the region* $\{0 < |z| < \frac{\pi}{2}\}$.

Method of long division : We see that $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$, by long division, we have

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + ...$$

The disadvantage is that we can not obtain the whole series. However, in this case, we do not have a closed form of Laurent series for $\frac{1}{\sin z}$.

Exercise:

1. Find the Laurent series of $\frac{z}{(z-1)(z-3)}$ in the region $\{0 < |z-1| < 2\}$.