THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 7
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0.1 Having Antiderivative and Analyticity

We may ask if having antiderivative are equivalent to analyticity. Having antiderivative must imply analyticity since we have $F'(z) = f(z)$ in which $F(z)$ is already analytic.

Theorem 1. *A function f that is analytic throughout a simply connected domain U must have an antiderivative everywhere in U and hence $\int_C f(z)dz = 0$ for any closed contour C lying in U.*

Remark : $f = 1/z$ gives a example here to verify that simply connectedness is necessary. You may consider $D = B_1(0) \setminus B_{1/2}(0)$ which is not simply connected but f is analytic here, however f do not have a antiderivative defined on D.

Theorem 2. *(Morera's theorem) A continuous f defined in open connected domain D such that $\int_C f(z)dz = 0$ for any closed contour C lying in D, then $f(z)$ is analytic in D.*

Remark : Actually the statement remains valid for any triangular path C lying in U, we will discuss the reason later.

0.2 Cauchy Integral Formula

Theorem 3. *(Cauchy Integral Formula) Let f be analytic inside and on a simple closed contour C. If z_0 is interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}.$$

Remark : You can see that an analytic function is uniquely determined by its boundary value. (compare with the case of real variable function)

Lemma 1. *Let h be continuous on a simple closed contour C. Define $H_n(z) = \int_C \frac{h(w)dw}{(w-z)^n}$ for $n \geq 1$ and z being inside the interior of C. Then H_n is analytic inside the interior of C and $H_n'(z) = nH_{n+1}(z)$.*

Using this lemma, we have:

Theorem 4. *(Generalized Cauchy Integral Formula) Let f be analytic inside and on a simple closed contour C. If z_0 is interior to C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

Remark : This is why analyticity implies complex infinite differentiability.

0.3 Some applications of Cauchy Integral Formula

Theorem 5. *(Cauchy's estimate) Suppose that a function f is analytic inside and on a positively oriented circle $C_R = \{z \in \mathbb{C} \mid |z - z_0| = R\}$. If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}.$$

Remark : It is a immediate consequence of generalized cauchy integral formula.
Remark : The maximum value M_R must exist since C_R is compact and f is analytic (hence continuous).

Theorem 6. *(Liouville's theorem) If f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.*

Remark : The proof is easy using Cauchy's estimate. If f is bounded, then the constant $M_R = M$ is independent of R. We have $|f'(z_0)| \leq \frac{M}{R}$ for any z_0 and $R > 0$, by taking $R \to \infty$, we have $f'(z_0) = 0$. Hence f is constant.

Remark : An important consequence is that non-constant entire function can not be bounded !(compare to real variable function) Since entire function must be bounded on compact set, so entire function becomes infinite at infinite. (Unless it is a constant function)

Theorem 7. *(Fundamental Theorem of Algebra) If $p(z)$ is non-constant polynomial, then there is a complex number a with $p(a) = 0$.*

Proof. We prove by contradiction. Suppose there is no $a \in \mathbb{C}$ such that $p(a) = 0$. Thus $p(z) \neq 0$ in $\mathbb{C}$, then $f = p^{-1}$ is entire. Suppose

$$p = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = z^n(a_0 z^{-n} + a_1 z^{-(n-1)} + ... + a_n),$$

where $a_n \neq 0$. Thus $\lim_{z\to\infty} p = \infty$ which implies $\lim_{z\to\infty} f = 0$. Since f is entire, then it must be continuous. We can find a large $R > 0$ such that $|f(z)| < 1$ if $|z| > R$. Since f is continuous on $\overline{B_R(0)}$, then it is bounded in $\overline{B_R(0)}$, says, $|f(z)| < M$ if $|z| \leq R$. Hence f is bounded thereofre by Liouville's theorem, $f = p^{-1}$ is constant, which contradicts to our assumption. □

Remark : It is a very short proof of Fundamental Theorem of Algebra by using complex analysis. The proof will be very long and hard if we use algebric method. (MATH3040 will introduce this proof)

Theorem 8. *(Maximum Modulus principle) Suppose f is analytic in a open connected domain Ω and $|f(z)| \leq |f(z_0)|$ at each point $z \in \Omega$. Then $f(z) = f(z_0)$ is constant throughout Ω.*

Remark : It is equivalent to say that if f is non-constant analytic function, then there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all z in the domain.

Remark : Under the assumption of this theorem, we can say the maximum value must appear on the boundary of the domain if the function is continuous up to boundary.

0.4 Exercise:

1. Let $f = \sum_0^\infty a_n z^n$ (supposed that the series is uniformly convergent) be entire such that $|f(z)| \leq A|z|$ for all z, where A is fixed constant. Show that $f = az$ where a is a constant.
2. Let $f = u + iv$ be entire and $u \leq M$ in $\mathbb{C}$, then f must be constant. (Hint: Consider e^f)
3. Let f be non-constant analytic in open connected U. Suppose $f \neq 0$ in $\overline{U}$, prove that $|f|$ can not attain its minimum value in U.
4. Let $f = e^z$ in the rectangle region R with $0 \leq x \leq 1$, $0 \leq y \leq \pi$, find the maximum and minimum value of $\text{Re}(f)$ in R.