THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 3
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Definition 1. *The* ***derivative*** *of a complex-valued function f at z_0 is the limit*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{(}h\text{ is non-zero complex number)}$$

and f is said to be ***differentiable*** *at z_0 when this limit exists. The derivative is denoted by $f'(z)$ or $\frac{d}{dz} f(z)$.*

Remark: The definition is equivalent to say that the limits of real and imaginary part of $\frac{f(z)-f(z_0)}{z-z_0}$ exist.

Proposition 1. *(Property of differentiation)*

- $\frac{d}{dz} z^n = n z^{n-1}$ *if n is an integer ($z \neq 0$ if the integer is negative)*
- $\frac{d}{dz} [cf(z)] = cf'(z)$ *if $c \in \mathbb{C}$ is a constant*
- $\frac{d}{dz} [f(z) + g(z)] = f'(z) + g'(z)$
- $\frac{d}{dz} [f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$
- *Let $F(z) = g(f(z))$ and $w = f(z)$, then $F'(z) = g'(w)f'(z)$.*

Theorem 1. *Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v exist at (x_0, y_0) and they satisfy the* ***Cauchy-Riemann equations***

$$u_x = v_y \ , \ u_y = -v_x.$$

And we can write $f'(z_0) = u_x + iv_x$.

Proof. (Sketch) By the definition of derivative,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

where h is a non-zero complex number. By definition of limit, the limiting values are the same if h approaches 0 by any paths. If we take $h = r$ and $h = ir$ for non-zero real number r respectively, then by comparing the real and imaginary parts of the limits we obtain the results. □

The 'converse' is also true:

Theorem 2. *Suppose that $f(z) = u(x, y) + iv(x, y)$. Suppose u and v have continuous first-order partial derivatives at (x_0, y_0). If u and v satisfy the Cauchy-Riemann equations*

$$u_x = v_y \ , \ u_y = -v_x,$$

then f is differentiable at (x_0, y_0).

Remark : The Cauchy-Riemann equations in polar coordinate is given by $ru_r = v_\theta$ and $u_\theta = -rv_r$. And we have $f'(z) = e^{-i\theta}(u_r + iv_r)$.

Remark : The continuity of the first-order partial derivatives is required here.(try the example $f = \frac{\bar{z}^2}{z}$)

Definition 2. *A complex function f is **analytic(holomorphic)** at a point z_0 if it is differentiable in $B_r(z_0)$ for some $r > 0$. f is **analytic(holomorphic)** in an open set U if it is analytic at each point in U.*

Remark : If a function is differentiable only at a point, then it is not analytic at that point. Analyticity must be defined in a open set (a neighborhood of a point).

Theorem 3. *If a complex function f is analytic at a point z_0, then it can be differentiated infinitely many times at that point.*

Definition 3. *An **entire** function is a function that is analytic at each point in the entire complex plane.*

$f = u + iv$ is said to be real differentiable if $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is real differentiable with $f(x, y) = (u(x, y), v(x, y))$. From Cauchy-Riemann equations we know that complex differentiability implies real differentiability. But the converse is not true, complex differentiability is stronger than real differentiability. The condition for real differentiability does not give us the Cauchy-Riemann equations.

Example 1. *(differentiable only at a point and thus not analytic)*

$$f(z) = |z|^2 = u + iv = x^2 + y^2.$$

It gives us $u_x = 2x$ and $v_y = 0$.

Example 2. *(does not satisfy Cauchy-Riemann equations and the limit f' does not exist but it is real differentiable)*

$$f(z) = u + iv = \overline{z} = x - iy.$$

Example 3. *(does not have continuous first-order partial derivatives but it satisfies the Cauchy-Riemann equations)*

$$f = \frac{\overline{z}^2}{z} \text{ for } z \neq 0, \;\; f = 0 \text{ for } z = 0.$$

Exercise :

1. Prove the Cauchy-Riemann equations in polar form.

2. Determine where $f'(z)$ exists and find its value (a) $f = z^2 + iy^2$,(b)$f = zlm(z)$.

3. Given $f = u + iv$ and $u = xy$, find v such that f is analytic.