THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 11
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0.1 Residue Theory

Definition 1. *Suppose that f is analytic in some punctured disk $D = \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. The coefficient of $\frac{1}{(z-z_0)}$ in the Larrent series is called the residue of f at the singular point $z = z_0$, which is denoted by $\operatorname*{Res}_{z=z_0} f$. If we write $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$, then $\operatorname*{Res}_{z=z_0} f = b_1$.*

Theorem 1. *(Cauchy Residue Theorem) Suppose C is a closed contour in positive sense. If f is analytic inside and on C except finite number of singular points z_k inside C, then*

$$\int_C f dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f$$

Remark : Actually it is exactly Cauchy integral formula in the view of power series.
Remark : In other words, to calculate the integral $\int_C f dz$ is to calculate the residue of f at the singular points.

When it comes to the computation of residue, of course we can express the whole Larrent series to obtain that. We provide an alternative method here. If the order of pole of f at $z = z_0$ is m and thus

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n}.$$

We consider

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^{n+m} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + ... + b_m$$

and differentiate it $m-1$ times, we could have

$$\frac{d^{m-1}}{dz^{m-1}}\Big[(z-z_0)^m f(z)\Big] = (m-1)!b_1 + O(z-z_0)$$

Theorem 2. *Suppose that f is analytic in some punctured disk $D = \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$ and the order of pole at z_0 is m, then*

$$\operatorname*{Res}_{z=z_k} f = \lim_{z\to z_0} \frac{1}{(m-1)!}\left(\frac{d^{m-1}}{dz^{m-1}}\Big[(z-z_0)^m f(z)\Big]\right)$$

Remark : You must know the order of the pole before using this theorem !
Remark : This method can not be used when the the singularity is an essential singularity.

Theorem 3 (Jordan's lemma)**.** *Consider a semicircle C_R represented by $z = Re^{i\theta}$ with $\theta \in [0, \pi]$ and an analytic function f in the upper-half plane. If $M_R = \max\big\{|f(z)|\big|z \in C_R\big\}$, then*

$$\left|\int_{C_R} f(z)e^{iaz}dz\right| \leq \frac{\pi M_R}{a}$$

Remark : In many applications, we usually have the condition that $M_R \to 0$ as $R \to \infty$, then we obtain that $\left|\int_{C_R} f(z)e^{iaz}dz\right| \to 0$ as $R \to \infty$.

0.2 Exercise:

1. Compute $\displaystyle\int_0^\infty \frac{1}{1+x^3}$.

2. Compute $\displaystyle\int_0^\infty \frac{(\log x)^2}{1+x^2}$.

3. Compute $\displaystyle\int_0^\pi \frac{1}{a^2+\sin^2\theta}$ for a non-zero.

4. Compute $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x^2+4x+5}$.