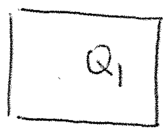
Lecture 12 and 13.

- Continued from last lecture. To prove the existence of anti-derivative function, we can restrict the integral curves to a smaller class, that is zig-zag curve. Therefore if we can show for all square in Ω, where Ω is a simply connected domain, ~~whe~~ we have $\int_{\text{bdry of squares}} f(z)\,dz = 0$.

Then f still has anti-derivative.

- We now consider Cauchy - Gousat.

<u>Version 1</u>:



f is analytic in $\overline{Q_1}$

then $\int_{\partial Q_1} f(z)\,dz = 0$.

<u>Pf</u>: Step 1: By dyadic decomposition, we have a sequence of $\{Q_n\}$ which satisfies

i) Length of Q_n, denoted by L_n, satisfies $L_n = \frac{1}{2} L_{n-1}$.

ii) $I_n = \int_{\partial Q_n} f(z)\,dz$ satisfies $|I_n| \geq \frac{1}{4} |I_{n-1}|$.

(iii) $\quad Q_n \subseteq Q_{n-1}$ s.t $\quad Q_n \longrightarrow z_0 \in \overline{Q_1}$

Step 2: $\quad z_0$ is analytic pt of f.

$\therefore \ \forall \varepsilon > 0. \ \exists \delta > 0$ s.t $\quad \left| \dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \right| < \varepsilon.$

if $\quad |z - z_0| < \delta.$

By (iii) in Step 1: we choose $n > n_0(\delta)$ large.

s.t $\quad |z - z_0| < \delta. \quad \forall z \in Q_n$ with $n > n_0(\delta)$.

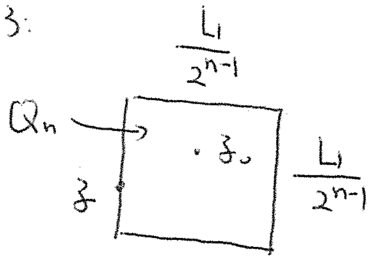
$$\therefore \ I_n = \int_{\partial Q_n} f(z)\, dz = \int_{\partial Q_n} f(z) - f(z_0) - f'(z_0)(z - z_0)\, dz$$

and can be bounded by

$$|I_n| \leq \int_{\partial Q_n} \left| f(z) - f(z_0) - f'(z_0)(z-z_0) \right| |dz|$$

$$\leq \varepsilon \int_{\partial Q_n} |z - z_0| \cdot |dz|$$

Step 3:



$$|z - z_0| \leq \underbrace{\sqrt{2} \cdot \frac{L_1}{2^{n-1}}}_{\text{Length of diagonal}}.$$

by step2.
$\Longrightarrow$

$$|I_n| \le \varepsilon \cdot \sqrt{2} \cdot \frac{L_1}{2^{n-1}} \cdot 4 \cdot \frac{L_1}{2^{n-1}}$$

By (ii) in step 2.

$$|I_1| \le 4^{n-1} |I_n| \le 4^{n-1} \cdot \varepsilon \sqrt{2} \cdot \frac{L_1}{2^{n-1}} \cdot 4 \cdot \frac{L_1}{2^{n-1}}$$

$$= 4\sqrt{2} L_1^2 \cdot \varepsilon$$

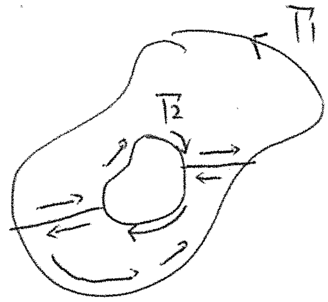
$\Rightarrow$ $I_1 = 0$ since ε can be arbitrarily small

Version 2. Simply connected domain case.

By arguments at the beginning and square version of Cauchy - Gousat, then f has anti-derivative function in Ω if Ω is simply connected and f is analytic in Ω. Therefore for all closed curve in Ω.

$$\int_{\substack{\text{closed} \\ \text{curves}}} f(z)\, dz = 0$$

Version 3: multiple connected domain



you may cut multiple Connected domain into several Simply Connected domains.

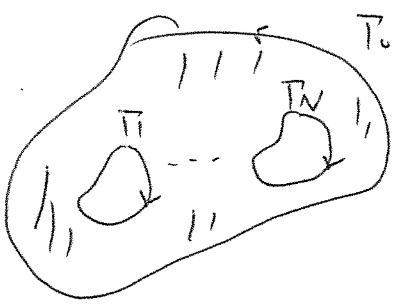
Apply Cauchy-Goursat for simply connected domains to each sub-domains above, then you have

$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0.$$

Notice 1: directions of Γ_1 and Γ_2 are different

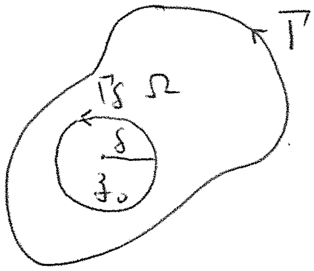
Notice 2: integrals on each common edges are cancelled.

$\therefore$ Generally



if f is analytic in shaded region then $\int_{\Gamma_0} + \cdots + \int_{\Gamma_N} f(z)\,dz = 0.$

· weak version of Cauchy-Gousat.



· f is analytic in $\overline{\Omega} \setminus \{z_0\}$

· $\lim_{z \to z_0} f(z)(z-z_0) = 0$.

then $\int_\Gamma f(z)\,dz = 0$.

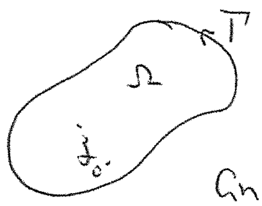
Pf: Cauchy-Gousat $\Rightarrow$ $\int_\Gamma f(z)\,dz = \int_{\Gamma_\delta} f(z)\,dz$.

Meanwhile $\left| \int_{\Gamma_\delta} f(z)\,dz \right| \le \int_{\Gamma_\delta} |f(z)(z-z_0)| \cdot \frac{1}{|z-z_0|} |dz|$

$\le \max_{z \in \Gamma_\delta} |f(z)(z-z_0)| \cdot \frac{1}{\delta} \cdot 2\pi\delta \to 0$ as $\delta \to 0$.

Cauchy integral formula.

Assumptions:



f is analytic in $\overline{\Omega}$.

and Ω is simply connected.

$\forall z_0$. define

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

$$\Longrightarrow \quad \int_\Gamma g(z)\,dz = \int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0. \quad \left(\text{weak Cauchy Goursat} \right)$$

$$\Longrightarrow \quad f(z_0)\int_\Gamma \frac{1}{z-z_0}\,dz = \int_\Gamma \frac{f(z)}{z-z_0}\,dz$$

By Cauchy - Goursat $\quad \int_\Gamma \frac{1}{z-z_0}\,dz = 2\pi i$

$$\Longrightarrow \quad f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,dz .$$

Rk1: an analytic function is uniquely determined by its bdry value.

Rk2: all derivatives of f exist.

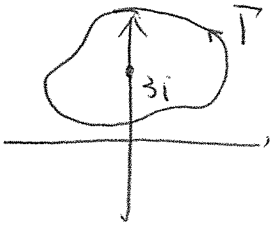
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz .$$

ex:

$$\int \frac{\cos z}{z(z^2+9)}\,dz .$$

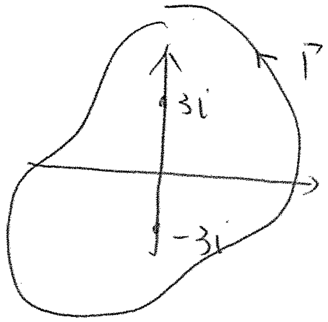
$0, 1$

ex:



$$\int_{\Gamma} \frac{\cos z}{z^2+9}\, dz .$$

ex.



$$\int_{\Gamma} \frac{\cos z}{z^2+9}\, dz .$$

ex:

$$\int \frac{e^{2z}}{z^4}\, dz .$$