lecture 5.

Thm: (i) at $z_0 = x_0 + iy_0$. $f = u + iv$ satisfies C-R equ.

(ii) $\partial_x u$, $\partial_y u$, $\partial_x v$ and $\partial_y v$ are continuous at (x_0, y_0).

then f is complexily derivable at z_0. Meanwhile.

$$f'(z_0) = \partial_x u\Big|_{x_0,y_0} + i\,\partial_x v\Big|_{x_0,y_0}.$$

ex

$$f(z) = x^2 - y^2 + 2xy \cdot i$$

$$\Rightarrow \quad f'(z) = 2x + 2yi = 2z.$$

ex.

$$f(z) = |z|^2.$$

f derivable only at $z=0$.

ex.

$$f(z) = \begin{cases} \bar{z}^2/z. & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

$\Rightarrow$ C-R is satisfied by f at $z=0$. but f is not complexily derivable at 0. you may check.

$\operatorname{Re} f = \dfrac{x^3 - 3xy}{x^2+y^2}$ has discontious derivables at 0.

C-R equ. by Polar coordinate.

$$\begin{cases} r u_r = u_\theta \\ u_\theta = -r v_r \end{cases} \quad \text{if} \quad f = u(r,\theta) + i v(r,\theta).$$

Moreover. $f'(z) = e^{-i\theta}(u_r + i v_r)$.

<u>ex</u>. $f(z) = \frac{1}{z^2} = \frac{1}{r^2 e^{2i\theta}}$

$\Rightarrow \quad f'(z) = -\frac{2}{z^3}$.

<u>ex</u>. $f(z) = z^n$ n is natural number

<u>ex</u> $f(z) = z^{1/2} \triangleq r^{1/2} e^{i\theta/2}$ $\theta \in (-\pi, \pi]$.

Def: f is analytic in D. where D is a connected open set $\Longleftrightarrow$ f is derivable at all pts in D.

Def. f is analytic at z_0 $\Longleftrightarrow$ $\exists$ D containing z. S.t. f is analytic on D. Here D is a connected open set

Def: f is entire $\Longleftrightarrow$ f is analytic in $\mathbb{C}$.

Def: $u(x,y)$ is analytic (real) in (x_0,y_0)

$$\Leftrightarrow \quad u(x,y) = u(x_0,y_0) + \partial_x u|_{x_0,y_0}(x-x_0) + \partial_y u|_{x_0,y_0}(y-y_0) + \cdots$$

holds in an open set containing (x_0,y_0)

i.e. Taylor Series $= u$ in an open set containing (x_0,y_0).

<u>Rk</u> $f = u+iv$ is analytic $\underset{\not\Leftarrow}{\Rightarrow}$ u, v are real analytic

<u>Rk</u> f is derivable at z_0 $\not\Leftrightarrow$ f is analytic at z_0.

Analytic condition is much more stronger than derivable condition

<u>ex</u> $f = |z|^2$.

0 is derivable but not analytic of f.