1.4 Relative Interior

Consider $I = [0, 1] \subset \mathbb{R}$. Then the interior of I is (0,1). However, if we consider I as a subset in $\mathbb{R}^2$, then the interior of I is empty. This motivates the following definition.

Definition:(Relative Interior) Let $C \subset \mathbb{R}^n$. We say that x is a *relative interior point* of C if $B(x;\epsilon)\cap \operatorname{aff}(C) \subset C$, for some $\epsilon > 0$. The set of all relative interior point of C is called the *relative interior* of C, and is denoted by $\operatorname{ri}(C)$. The *relative boundary* of C is equal to $\overline{C}\backslash\operatorname{ri}(C)$.

The following is the most fundamental result about relative interiors.
Proposition:(Line Segment Property) Let C be a nonempty convex set. If $x \in \operatorname{ri}(C)$, $\overline{x} \in \overline{C}$, then $\lambda x + (1-\lambda)\overline{x} \in \operatorname{ri}(C)$ for $\lambda \in (0, 1]$.

Proof. Fix $\lambda \in (0.1]$. Consider $x_\lambda = \lambda x + (1-\lambda)\overline{x}$.
Let L be the subspace parallel to $\operatorname{aff}(C)$. Define $\overline{B}(0,\epsilon) := \{z \in L|\ \|z\| < \epsilon\}$.
Since $\overline{x} \in \overline{C}$, for all $\epsilon > 0$, we have $\overline{x} \in C + \overline{B}(0,\epsilon)$. Then

$$\begin{aligned} B(x_\lambda;\epsilon) \cap \operatorname{aff}(C) &= \{\lambda x + (1-\lambda)\overline{x}\} + \overline{B}(0;\epsilon) \\ &\subset \{\lambda x\} + (1-\lambda)C + (2-\lambda)\overline{B}(0;\epsilon) \\ &= (1-\lambda)C + \lambda\left[x + \overline{B}\left(0; \frac{2-\lambda}{\lambda}\epsilon\right)\right] \end{aligned}$$

Since $x \in \operatorname{ri}(C)$, $x + \overline{B}\left(0; \frac{2-\lambda}{\lambda}\epsilon\right) \subset C$, for sufficiently small ϵ.
So $B(x_\lambda;\epsilon) \cap \operatorname{aff}(C) \subset \lambda C + (1-\lambda)C = C$ (since C is convex). Therefore, $x_\lambda \in \operatorname{ri}(C)$.

□

Proposition:(Prolongation Lemma) Let C be a nonempty convex set. Then we have

$$x \in \operatorname{ri}(C) \Longleftrightarrow \forall \overline{x} \in C,\ \exists \gamma > 0 \text{ such that } x + \gamma(x - \overline{x}) \in C.$$

In other words, x is a relative interior point iff every line segment in C having x as one of the endpoints can be prolonged beyond x without leaving C.

Proof. Suppose the condition holds for x. Let $\overline{x} \in \operatorname{ri}(C)$. If $x = \overline{x}$, then we are done. So assume $x \neq \overline{x}$. Then there exists $\gamma > 0$ such that $y = x + \gamma(x - \overline{x}) \in C$. Hence $x = \frac{1}{1+\gamma}y + \frac{\gamma}{1+\gamma}\overline{x}$. Since $\overline{x} \in \operatorname{ri}(C)$, $y \in C$, by the line segment property, we have $x \in \operatorname{ri}(C)$. The other direction is clear from the fact that $x \in \operatorname{ri}(C)$. □

Proposition: Let C be a nonempty convex set. Then

(a) $\overline{C} = \overline{\operatorname{ri}(C)}$.

(b) $\text{ri}(C) = \text{ri}(\overline{C})$.

(c) Let D be another nonempty convex set. Then the following are equivalent:

(i) C and D have the same relative interior.

(ii) C and D have the same closure.

(iii) $\text{ri}(C) \subseteq D \subseteq \overline{C}$.

Proof. (a) $\overline{\text{ri}(C)} \subset \overline{C}$ since $\text{ri}(C) \subset C$. Conversely, suppose $x \in \overline{C}$.
Let $\overline{x} \in \text{ri}(C)$. Consider $x_k = \frac{1}{k}\overline{x} + (1 - \frac{1}{k})x$. By the line segment property, each $x_k \in \text{ri}(C)$. Also, $x_k \to x$. Therefore, $x \in \text{cl}(\text{ri}(C))$.

(b) Note that $\text{aff}(C) = \text{aff}(\overline{C})$. Then by the definition of relative interior, $\text{ri}(C) \subseteq \text{ri}(\overline{C})$. Now suppose $\overline{x} \in \text{ri}(\overline{C})$, we will show that $\overline{x} \in \text{ri}(C)$.
Pick $x \in \text{ri}(C)$. We may assume $x \neq \overline{x}$.
Then by the prolongation lemma, there exists $\gamma > 0$ such that

$$\overline{x} + \gamma(\overline{x} - x) \in \overline{C}$$

Then by the line segment property and the fact that $x \in \text{ri}(C)$,

$$\overline{x} = \frac{\gamma}{\gamma + 1}x + \frac{1}{\gamma + 1}(\overline{x} + \gamma(\overline{x} - x) \in \text{ri}(C)$$

(c) Suppose $\text{ri}(C)$=$\text{ri}(D)$, then $\overline{\text{ri}(C)} = \overline{\text{ri}(D)}$. Hence $\overline{C} = \overline{D}$.
Suppose $\overline{C} = \overline{D}$, then $\text{ri}(C)$=$\text{ri}(\overline{C})$=$\text{ri}(\overline{D})$=$\text{ri}(D)$.
Therefore (i) and (ii) are equivalent.
Suppose $\overline{C} = \overline{D}$, then

$$\text{ri}(C) = \text{ri}(D) \subseteq D \subseteq \overline{D} = \overline{C}$$

Suppose $\text{ri}(C) \subseteq D \subseteq \text{cl}(C)$, then $\overline{\text{ri}(C)} \subseteq \overline{D} \subseteq \overline{C}$.
Since $\overline{\text{ri}(C)} = \overline{C}$, $\overline{\text{ri}(C)} = \overline{D} = \overline{\text{ri}(D)}$.
Hence $\overline{C} = \overline{D}$ and (ii),(iii) are equivalent.

□

1.5 Projection to Convex Sets

Given a set $C \subseteq \mathbb{R}^n$, the distance of a point to C is defined by

$$d(x; C) := \inf\{||x - y|| \mid y \in C\}$$

For closed convex sets, an important consequence is the following projection property.
Proposition:(Projection Property) Let C be a nonempty, closed convex subset of $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, there exists an unique $w \in C$ such that

$$||x - w|| = d(x; C)$$

w is called the projection of x to C, and is denoted by $P_C(x)$.

Proof. By definition of $d(x;C)$, there exists $w_k \in C$ such that

$$d(x;C) \leq ||x - w_k|| < d(x;C) + \frac{1}{k}$$

It follows that $\{w_k\}$ is a bounded sequence. Hence it has a converging subsequence $\{w_{k_l}\}$ which converges to a point w. Since C is closed, $w \in C$.
Considering the limit of

$$d(x;C) \leq ||x - w_{k_l}|| < d(x;C) + \frac{1}{k_l}$$

Hence $d(x;C) = ||x - w||$.
Now suppose $w_1 \neq w_2 \in C$ satisfy

$$||x - w_1|| = ||x - w_2|| = d(x;C)$$

Then we have,

$$2||x - w_1||^2 = ||x - w_1||^2 + ||x - w_2||^2 = 2||x - \frac{w_1 + w_2}{2}||^2 + \frac{||w_1 - w_2||^2}{2}$$

Since C is convex, $\frac{w_1+w_2}{2} \in C$. This gives,

$$||x - \frac{w_1 + w_2}{2}||^2 = ||x - w_1||^2 - \frac{||w_1 - w_2||^2}{4} < ||x - w_1||^2 = d(x;C)^2$$

But since C is convex, $\frac{w_1+w_2}{2} \in C$, this is a contradiction. □

Proposition: Let C be a nonempty, closed convex set, then $w = P_C(x)$ if and only if

$$\langle x - w, u - w \rangle \leq 0, \ \forall u \in C$$

Proof. Suppose $w = P_C(x)$.
Let $u \in C, \ \lambda \in (0,1)$. Since C is convex, $\lambda u + (1-\lambda)w \in C$. Then

$$||x-w||^2 = d(x;C)^2 \leq ||x-w-\lambda(u-w)||^2 = ||x-w||^2 - 2\lambda\langle x-w, u-w\rangle + \lambda^2||u-w||^2.$$

That is

$$2\langle x - w, u - w \rangle \leq \lambda||u - w||^2$$

Letting $\lambda \to 0^+$, we have

$$\langle x - w, u - w \rangle \leq 0$$

Conversely, suppose

$$\langle x - w, u - w \rangle \leq 0, \ \forall u \in C$$

Then

$$\begin{aligned} ||x - u||^2 &= ||x - w||^2 + 2\langle x - w, w - u \rangle + ||w - u||^2 \\ &\geq ||x - w||^2 - 2\langle x - w, u - w \rangle \geq ||x - w||^2 \end{aligned}$$

Hence $||x - w|| \leq ||x - u||$ for all $u \in C$ and $w = P_C(x)$. □

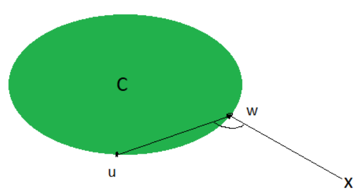

Figure 1: Projection to a convex set