1 Convex Sets and Functions

1.1 Convex Sets

Definition:(Convex sets) A subset C of $\mathbb{R}^n$ is called *convex* if

$$\lambda x + (1-\lambda)y \in C, \ \forall \ x, y \in C, \ \forall \lambda \in [0,1].$$

Geometrically, it just means that the line segment joining any two points in a convex set C lies in C.

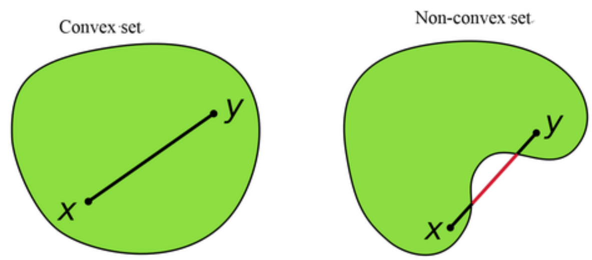

Figure 1: Examples of convex and non-convex set

Definition:(Convex combination) Given $x_1, ..., x_m \in \mathbb{R}^n$, an element in the form $x = \sum_{i=1}^m \lambda_i x_i$, where $\sum_{i=1}^m \lambda_i = 1$ and $\lambda_i \geq 0$ is called a convex combination of $x_1, ..., x_m$.

Proposition: A subset C of $\mathbb{R}^n$ is convex if and only if contains all convex combination of its element.

Proof. Suppose C is convex. We will show by induction that it contains all convex combination $\sum_{i=1}^m \lambda_i x_i$ of its elements.
The case $m = 1, 2$ is trivial, so suppose all convex combination of k elements lies in C, where $k \leq m$. Consider

$$x := \sum_{i=1}^{m+1} \lambda_i x_i, \text{ where } \sum_{i=1}^{m+1} \lambda_i = 1$$

If $\lambda_{m+1} = 1$, then $\lambda_1 = \cdot = \lambda_m = 0$. Then $x \in C$. So assume $\lambda_{m+1} < 1$, then

$$\sum_{i=1}^{m} \lambda_i = 1 - \lambda_{m+1} \text{ and } \sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_{m+1}} = 1$$

Then $y = \sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_{m+1}} x_i \in C$. Hence

$$x = (1 - \lambda_{m+1})y + \lambda_{m+1}x_{m+1} \in C$$

The other direction is trivial. □

Proposition: Let C_1 be a convex set of $\mathbb{R}^n$ and let C_2 be a convex set pf $\mathbb{R}^m$. Then the Cartesian product $C_1 \times C_2$ is a convex subset of $\mathbb{R}^n \times \mathbb{R}^m$.

1.1.1 Examples of Convex Sets

(a) Open and closed balls in $\mathbb{R}^n$.

(b) *Hyperplanes*: $\{x : \langle a, x\rangle = b,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}\}$.

(c) *Halfspaces*: $\{x : \langle a, x\rangle \le b,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}\}$.

(d) *Non-Negative Orthant*: $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \ge 0\}$.

(e) Convex cones: C is called a *cone* if $\alpha x \in C, \forall \alpha > 0,\ x \in C$. A cone which is convex is called a *convex cone*.

Proposition: Let $\{C_i \mid i \in \mathrm{I}\}$ be a collection of convex sets. Then:

(a) $\cap_{i \in I} C_i$ is convex, where each C_i is convex.

(b) $C_1 + C_2 = \{x + y \mid x \in C_1, y \in C_2\}$ is convex.

(c) λC is convex for any convex sets C and scalar λ. Furthermore, $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$ for positive λ_1, λ_2.

(d) C^{o}, $\overline{C}$ are convex, i.e. the interior and closure of a convex set are convex.

(e) $T(C)$, $T^{-1}(C)$ are convex, where T is a linear map.

Proof. Parts (a)-(c), (e) follows from the definition (Exercise!). Let's prove (d).
Interior Let $x, y \in C^{\mathrm{o}}$. Then there exists r such that balls with radius r centred at x and y are both inside C.
Suppose $\lambda \in [0, 1]$ and $||z|| < r$. By convexity of C, we have,

$$\lambda x + (1 - \lambda)y + z = \lambda(x + z) + (1 - \lambda)(y + z) \in C$$

Therefore, $\lambda x + (1 - \lambda)y \in C^{\mathrm{o}}$. Hence C^{o} is convex.
Closure Let $x, y \in \overline{C}$. Then there exists sequences $\{x_k\} \subset C, \{y_k\} \subset C$ such that $x_k \to x, y_k \to y$. Suppose $\alpha \in [0, 1]$. Then for each k,

$$\lambda x_k + (1 - \lambda)y_k \in C$$

But $\lambda x_k + (1 - \lambda)y_k \to \lambda x + (1 - \lambda)y \in \overline{C}$. Hence, $\overline{C}$ is convex. □

1.2 Convex and Affine Hulls

1.2.1 Convex Hull

Definition:(Convex Hull)
Let X be a subset of $\mathbb{R}^n$. The convex hull of X is defined by

$$\operatorname{conv}(X) := \bigcap \{C | \ C \text{ is convex and } X \subseteq C\}$$

In other words, $\operatorname{conv}(X)$ is the smallest convex set containing X.
The next proposition provides a good representation for elements in the convex hull.

Proposition: For any subset X of $\mathbb{R}^n$,

$$\operatorname{conv}(X) = \Big\{ \sum_{i=1}^{m} \lambda_i x_i | \ \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0,\ x_i \in X \Big\}$$

Proof. Let $C = \left\{ \sum_{i=1}^{m} \lambda_i x_i | \ \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0,\ x_i \in X \right\}$. Clearly, $X \subseteq C$. Next, we check that C is convex.
Let $a = \sum_{i=1}^{p} \alpha_i a_i, b = \sum_{j=1}^{q} \beta_j b_j$ be elements of C, where $a_i, b_i \in C$ with $\alpha_i, \beta_j \geq 0$ and $\sum \alpha_i = \sum \beta_j = 1$. Suppose $\lambda \in [0,1]$, then

$$\lambda a + (1-\lambda) b = \sum_{i=1}^{p} \lambda \alpha_i a_i + \sum_{j=1}^{q} (1-\lambda) \beta_j b_j.$$

Since

$$\sum_{i=1}^{p} \lambda \alpha_i + \sum_{j=1}^{q} (1-\lambda) \beta_j = \lambda \sum_{i=1}^{p} \alpha_i + (1-\lambda) \sum_{j=1}^{q} \beta_j = 1$$

we have $\lambda a + (1-\lambda) b \in C$. Hence, C is convex. Also, $\operatorname{conv}(X) \subseteq C$ by the definition of $\operatorname{conv}(X)$.
Suppose $a = \sum \lambda_i a_i \in C$. Then since each $a_i \in X \subseteq \operatorname{conv}(X)$ and $\operatorname{conv}(X)$ is convex, we have $a \in \operatorname{conv}(X)$. Therefore, $\operatorname{conv}(X) = C$. $\square$