Let G be a Kac-Moody Group. Our goal today would be ot discuss the topological properties on the non-negative half $G_{\geq 0}$ (some of our results will extend to any semifield $\Bbbk$, but we often require to consider only $\Bbbk = \mathbb{Z}$ for the necessary topological properties).

6.1 Setup

Let G be a Kac-Moody Group (one could consider GL_n), W its Weyl group and by definition,

$$\begin{aligned} U^{\pm}_{\geq 0} &= \bigsqcup_{w \in W} U^{\pm}_{w,>0} \\ G_{\geq 0} &= \bigsqcup_{u,v \in W} G_{u,v,>0} = \bigsqcup_{u,v \in W} U^{+}_{u,>0} T_{>0} U^{-}_{v,>0}. \end{aligned} \tag{1}$$

Recall that from the previous lecture, we have the bijection of sets

$$\begin{aligned} U^{\pm}_{w,\geq 0} &= \mathbb{R}^{\ell(w)}_{>0} \\ G_{u,v,>0} &= \mathbb{R}^{\ell(u)-\ell(v)+\mathrm{rk}(G)}_{>0}. \end{aligned}$$

where u, v, w are elements of W and ℓ is the length function of W (in the sense of a Coxeter Group). In this lecture, we shall upgrade this bijection of sets into a homeomorphism.
Furthermore, we shall show that Equation 1 is also a cellular decomposition (i.e. the Hausdorff closure of each cell ($\mathbb{R}^k$) is a union of cells. So $\overline{U^{\pm}_{w,>0}}$ is the union of cells of the form $U^{\pm}_{v,>0}$.)

We use $G^{\pm}$ as a short hand for G^{+} or G^{-} (and similarly for U), do not be confused with the introduction of a new symbol.

6.2 Example

Let us first compute an example first. For this subsection only, let $G = \mathrm{GL}_3$.
Then,

$$\begin{array}{ccccc} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix} \middle| \ \alpha, \beta, \gamma \in \mathbb{R} \right\} & = & U^{-} & \xrightarrow{\sim} & \mathbb{R}^3 \\ & & \uparrow & & \\ \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \alpha\gamma - \beta > 0\} & = & U^{-}_{\geq 0} & \xrightarrow{\sim} & \mathbb{R}^3_{\geq 0} \end{array} \tag{2}$$

One can see the second equality in two different ways. We could see that the Weyl Group of G is of type A_2, which is finite so, having the longest word w_0 with a minimal length representation $c_1 c_2 c_1$,

$$(U^{-}_{\geq 0})^{\circ} = U^{-}_{w_0,>0} = \{y_1(a) y_2(b) y_1(c) \mid a, b, c \in \mathbb{R}_{>0}\}. \tag{3}$$

Alternatively, one can recall the characterization of Totally Positive Matrices from Lecture 1, thus the submatrix $\begin{pmatrix} \alpha & 1 \\ \beta & \gamma \end{pmatrix}$ must have positive determinant (positive minor). From this example, we can also see that the closure relation is non-trivial.

We have that

$$\begin{aligned} U^-_{\geq 0} - U^-_{>0} &= U^-_{e,>0} \sqcup U^-_{s_1,>0} \sqcup U^-_{s_2,>0} \sqcup U^-_{s_1 s_2,>0} \sqcup U^-_{s_2 s_1,>0} \\ &= \mathbb{R}^0_{>0} \sqcup \mathbb{R}^1_{>0} \sqcup \mathbb{R}^1_{>0} \sqcup \mathbb{R}^2_{>0} \sqcup \mathbb{R}^2_{>0} \end{aligned}$$

but

$$\mathbb{R}^3_{\geq 0} - \mathbb{R}^3_{>0} = \mathbb{R}^0_{>0} \sqcup \mathbb{R}^1_{>} \sqcup \mathbb{R}^1_{>0} \sqcup \mathbb{R}^1_{>0} \sqcup \mathbb{R}^2_{>0} \sqcup \mathbb{R}^2_{>0} \sqcup \mathbb{R}^2_{>0}.$$

6.3 Results

Lemma 6.1. *The non-negative half $U^{\pm}_{\geq 0}$ is closed in $U^{\pm}$.*

Proof. Without loss of generality, we shall consider the sets $U^-_{\geq 0}$ and U^-.
Our strategy is to show that the following composition of maps is proper. If the following composition of maps is proper, then the map β has to be closed, showing our result.

$$(\mathbb{R}_{\geq 0})^N \xrightarrow{\alpha} U^-_{\geq 0} \xhookrightarrow{\beta} U^- \xrightarrow{\gamma} \tfrac{U^-}{[U^-,U^-]} \cong \mathbb{R}^m$$

In order to define this map, we have to let $s_{d_1} s_{d_2} ... s_{d_N}$ be a reduced expression of the longest word. Then, we let an element $(a_1, a_2, ..., a_N) \in (\mathbb{R}_{\geq 0})^N$ be sent to $\prod_{i=1}^N y_{d_i}(a_i) \in U^-_{\geq 0}$. ($\prod_{i=1}^N c_i$ is notational shorthand for $c_1 c_2 ... c_N$ noting that the order is important.)
But any expression of the form $\gamma \circ \beta(\prod_{i=1}^N y_{\alpha_i}(a_i))$ is a m-dimensional vector with its i-coordinate $\sum_{1 \leq j \leq N, d_i = d_j} a_j$. Hence, for any compact rectangle of the form

$$I = [0, b_1] \times [0, b_2] \times ... \times [0, b_m] \in \mathbb{R}^m$$

the pre-image under our overall map is

$$(\gamma \circ \beta \circ \alpha)^{-1}(I) = \left\{ (a_1, a_2, ..., a_N) \in (\mathbb{R}_{\geq 0})^N \;\middle|\; \sum_{1 \leq j \leq N, d_i = d_j} a_j \leq b_i \right\}.$$

This is compact in $(\mathbb{R}_{\geq 0})^N$. □

Example 6.2. *Let us be explicit about the map α and γ in the case of $G = \mathrm{GL}_3$. In this case, for $(a_1, a_2, a_3) \in (\mathbb{R}_{\geq 0})^3$ and the longest word to be of the form $c_1 c_2 c_1$, where $y_1(a)$ corresponds to the matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the maps α and γ are

$$\alpha(a_1, a_2, a_3) = \begin{pmatrix} 1 & 0 & 0 \\ a_1 + a_2 & 1 & 0 \\ a_1 a_2 & a_2 & 1 \end{pmatrix}$$

$$\gamma \begin{pmatrix} 1 & 0 & 0 \\ a_1 + a_2 & 1 & 0 \\ a_1 a_2 & a_2 & 1 \end{pmatrix} = (a_1 + a_3, a_2).$$

Next, we summarize various properties of Lusztig's Canonical Basis.

Theorem 6.3. *Let G be a simply-laced Group and $V \in \mathrm{Irr}_{f.\ d.}\mathbb{C}[G]$ be an irreducible finite-dimensional complex representation. Then,*

a) $\mathrm{End}(V)$ *is a matrix group, with a fixed basis set S.*

b) If $g \in G_{\geq 0}$, then there exists $\tilde{g} \in \mathrm{End}(V)$ such that the (i,j)-th entry $(\tilde{g})_{ij} \geq 0$ and $\prod_i (\tilde{g})_{ii} \geq 1$ when using the basis S.

c) Furthermore, as in (b), if $g \in G_{>0}$, then there exists $\tilde{g} \in \mathrm{End}(V)$ such that the (i,j)-th entry $(\tilde{g})_{ij} > 0$ when using the basis S.

d) Let w be a highest weight vector, then w is an element of S.

Proof. See [Bump & Schilling's Crystal Bases: Representations And Combinatorics §2.2, 15.3]. □

Remark 6.4. *Here, $G_{>0}$ means the set $U^+_{>0}T_{>0}U^-_{>0}$.*

Example 6.5. *Let us consider $G = \mathrm{SL}_n$ with $V \cong \mathbb{C}^n$ the standard representation. We can write an element $g \in G_{\geq 0}$ having the form in $\mathrm{End}(V) = \mathrm{Mat}_{n\times n}(\mathbb{C})$,*

$$\tilde{g} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ * & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & a_n \end{pmatrix} \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 + b_1 & * & \dots & * \\ * & a_2 + b_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & a_n + b_n \end{pmatrix},$$

where $a_i, b_i \in \mathbb{R}_{\geq 0}$. But, we have the determinant $\prod_i a_i = 1$, showing that $\prod_i (a_i + b_i) \geq 1$.

Theorem 6.6. *The non-negative part $G_{\geq 0}$ is closed in G.*

Proof. Let us first do it for the case where G is simply-laced.
Part 1: The closure of $G_{\geq 0}$ is in B^-B^+.
A way to characterize an element $g \in G$ to be contained in B^-B^+ is to say that whenever we have $V \in \mathrm{Irr}_{\mathrm{f.\ d.}}\mathbb{C}[G]$ to be a faithful irreducible finite-dimensional complex representation, with highest weight vector λ, then the corresponding $\tilde{g} \in \mathrm{End}(V)$ satisfies $\tilde{g}\lambda \neq 0$. Using Theorem 6.3, we can see that any such g would satisfy $\prod_i \tilde{g}_{ii} > 1$, and hence any g' in the closure of $G_{>0}$ would satisfy $\prod_i \tilde{g'}_{ii} \geq 1$, which implies that $\tilde{g'}\lambda \neq 0$. Hence, the closure has to lie in B^-B^+.
Part 2: Consider the following diagram

$$\begin{array}{ccc} B^-B^+ & = & U^- \times T \times U^+ \\ & & \alpha\uparrow \quad \beta\uparrow \quad \gamma\uparrow \\ G_{\geq 0} & = & U^-_{\geq 0} \times T_{>0} \times U^+_{\geq 0} \end{array} \tag{4}$$

Here, the maps α and γ are closed by Theorem 6.1, while the map β is simply the identity component included into T. Hence, this concludes the case where G is simply-laced
If G is not simply-laced, we can use folding. Let $\hat{G}$ be a simply-laced group with a diagram automorphism $\sigma : \hat{G} \to \hat{G}$ where $(\hat{G})^\sigma = G$. By definition, $G_{\geq 0} = (\hat{G}_{\geq 0})^\sigma = (\hat{G}_{\geq 0}) \cap \hat{G}^\sigma = (\hat{G}_{\geq 0}) \cap G$ is closed in G. □

The next lemma shows the cellular decomposition of cells.

Lemma 6.7. *Let W be the Weyl group, $\leq$ be the Bruhat order on the Weyl Group and $w \in W$. Then,*

$$\overline{U^-_{w,>0}} = \bigsqcup_{w' \leq w} U^-_{w',>0}$$

.

Proof. First, we shall show $\overline{U^-_{w,>0}} \supseteq \bigsqcup_{w' \leq w} U^-_{w',>0}$. Let, $c_1c_2...c_k$ be a reduced expression for w. Then, there exists an expression of any element $w' \leq w$ as $c_{d_1}c_{d_2}...c_{d_r}$ where $1 \leq d_1 < d_2 < ... < d_r \leq k$. Thus, as $\lim_{a_k \to 0} y_k(a_k) = 1$, we can consider the set $\{\lim_{a_j \to 0 \forall j \in J} \prod_i y_i(a_i)\}$ where $J = \{1, 2, ..., k\} \setminus \{d_1, d_2, ..., d_r\}$.
Conversely, we have

$$\overline{U^-_{w,>0}} \subseteq \overline{B^+ w B^+} \tag{5}$$

$$= \bigsqcup_{w' \leq w} B^+ w' B^+ \tag{6}$$

and as $U^- \geq 0$ is closed in U^- and thus closed in G,

$$\overline{U^-_{w,>0}} \subseteq \bigsqcup_{w' \leq w} (B^+ w' B^+ \cap U^-_{\geq 0}) \tag{7}$$

$$= \bigsqcup_{w' \leq w} U^-_{w',>0}. \tag{8}$$

□

Finally, we shall see that this are indeed "cells".

Theorem 6.8. *Let W be the Weyl group and $w \in W$. Then, we have the homeomorphism $U^-_{w,>0} \cong (\mathbb{R}_{\geq 0})^{\ell(w)}$.*

Proof. We rely on the Invariance of Domain: let U is an open set in $\mathbb{R}^n$ and $f : U \to R^n$ be a continuous injective map, then $f(U)$ is open and f give us an homeomorphism between U and its image $f(U)$.
Consider the following map

$$f : (\mathbb{R}_{\geq 0})^{\ell(w)} \times (\mathbb{R}_{\geq 0})^{N-\ell(w)} \to U^-_{w,>0} \times U^-_{w^{-1}w_0,>0} \cong U^-_{w_0,>0} \cong U^-_{>0}$$

by sending $(a_1, a_2, ..., a_{\ell(w)}) \times (a_{\ell(w)+1}, ..., a_N)$ to $\prod_{i=1}^{\ell(w)} y_i(a_i) \times \prod_{i=\ell(w)+1}^{N} y_i(a_i)$. Here, $c_1c_2...c_{\ell(w)}$ is a reduced expression for w and $c_1c_2...c_N$ is a reduced expression for w_0, the longest element.
When, $w = e$, this is a continuous injective map, and thus by Invariance of Domain, shows the case where $w = w_0$. This in turns shows the general case, as f is then an homeomorphism between the domain and the image, and thus, the components of the products are also homeomorphisms. □