16 2 2022

Last week. Construction of $U^-_{\geq 0}$ (for GL_n, or more generally, for simply Laced gps)

"Simply Laced" means that for ~~every~~ any vertices in the Dynkin diagram,

either ($\circ \quad \circ$) or ($\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$)

Dynkin diagram for GL_n: $\circ\!-\!\circ \cdots \circ\!-\!\circ$ (type A_n)

for SO_{2n} (type D_n): $\circ\!-\!\circ \cdots \circ\!<$

$U^-_{\geq 0}$ is the monoid gen. by $y_i(a)$ for $i \in I$, $a > 0$, subject to the relations:

- $y_i(a) y_i(b) = y_i(a+b)$
- Coxeter relation $y_i(a) y_j(b) = y_j(b) y_i(a)$ $\quad \overset{i}{\circ} \quad \overset{j}{\circ}$

$$y_i(a) y_j(b) y_i(c) = y_j\left(\frac{bc}{a+c}\right) y_i(a+c) y_j\left(\frac{ab}{a+c}\right) \quad \overset{i}{\circ}\!-\!\overset{j}{\circ}$$

Consequences: ① $\underset{\underset{\mathbb{R}^{\ell(w)}_{>0}}{\wr\wr}}{U^-_{w,>0}}$ is well-defined (i.e. indep. of the choice of reduced expressions)

(for simply laced gps) ② $U^-_{\geq 0} = \coprod_{v \in W} U^-_{w,>0}$

Rank 2 Kac-Moody gp:

Dynkin diagram:	$\circ \quad \circ$	$\circ\!-\!\circ$	$\circ\!\Rightarrow\!\circ$	$\circ\!\Rrightarrow\!\circ$	$\circ\overset{\infty}{-}\circ$
Group:	$A_1 \times A_1$	A_2	B_2/C_2	G_2	$\widetilde{A}_1$

Coxeter relation (for rank 2): $\circ \quad \circ$ ✓ $\qquad \circ\!-\!\circ$ ✓

$\circ\overset{\infty}{-}\circ$ no Cox. relation.

$\overset{s}{\circ}\overset{\infty}{-}\overset{t}{\circ}$ $\quad (st)^{m_{st}} = 1$, i.e. $\underbrace{st\cdots}_{m_{st}} = \underbrace{ts\cdots}_{m_{st}}$

Here $m_{st} = \infty$, by convention, there is no extra relation.

$W = \langle s,t \rangle / s^2 = t^2 = 1$

$y_s(a_1) y_t(a_2) y_s(a_3) \cdots\cdots \in B^+(sts\cdots)B^+$

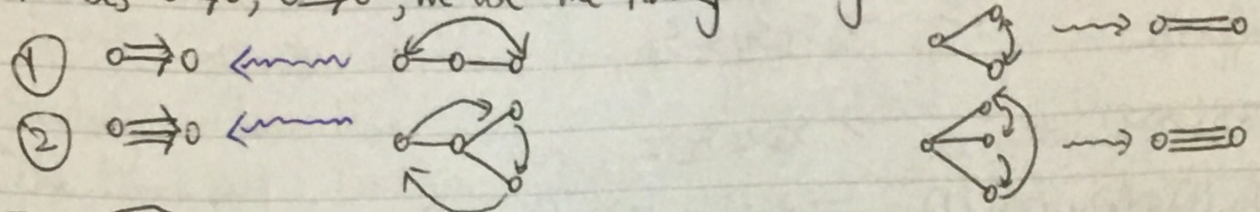
$y_t(b_1) y_s(b_2) y_t(b_3) \cdots\cdots \in B^+(tst\cdots)B^+$

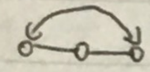
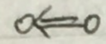
In this case, $sts\cdots \neq tst\cdots$

So $B^+(sts\cdots)B^+ \cap B^+(tst\cdots)B^+ = \phi$ $\qquad$ No Cox relation

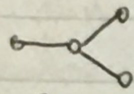
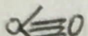
So in this case, $U^-_{\geq 0} = \langle y_s(a), y_t(b) \rangle \Big/ \begin{matrix} y_s(a_1) y_s(a_2) = y_s(a_1+a_2) \\ y_t(b_1) y_t(b_2) = y_t(b_1+b_2) \end{matrix}$

For cases ⇒, ⇛, we use the ~~flowing~~ "folding method".

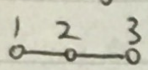
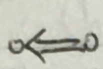


①  

SL_4 $\qquad$ $Sp_4 = SL_4^\sigma$

②  

SO_8 $\qquad$ $G_2 = SO_8^\sigma$

II I

$m_{st}=4$

$s_I s_{II} s_I s_{II} = s_{II} s_I s_{II} s_I$ is the shadow of

$w := s_1 s_3 s_2 s_1 s_3 s_2 = s_2 s_1 s_3 s_2 s_1 s_3$

Here $s_I = s_1 s_3$, $s_{II} = s_2$. both reduced expressions of type A_3.

Thus $y_I(a)\, y_{II}(b)\, y_I(c)\, y_{II}(d) = y_{II}(a')\, y_I(b')\, y_{II}(c')\, y_I(d')$.

Coxeter relation for type B_2/C_2 comes from the two expressions of U_{w_0} in A_3.

So from the result in A_3, we have that

$$(a,b,c,d) \longmapsto (a',b',c',d') \text{ is bijection on } \mathbb{R}^4_{>0}$$

Similar result holds for G_2.

Fact: Let (W,S) be a Coxeter group and σ is a diagram automorphism on W (i.e. a group hom. with $\sigma(S)=S$). Then W^σ is a Cox gp with the set of generators $\{w_J\}$, where J runs over σ-orbits on S such W_J is finite and w_J is the unique longest element of W_J.

Moreover, for $w_1, w_2 \in W^\sigma \subseteq W$,

$$l_\sigma(w_1 w_2) = l_\sigma(w_1) + l_\sigma(w_2) \text{ iff } l(w_1 w_2) = l(w_1) + l(w_2)$$

In our case . $W = S_4$, $S = \{1,2,3\}$, $\sigma(1)=3, \sigma(2)=2$, $\sigma(3)=1$

W^σ is generated by $\underset{\substack{\parallel \\ s_I}}{s_1 s_3}$, $\underset{\substack{\parallel \\ s_{II}}}{s_2}$ $\qquad$ $J = \{1,3\}$ or $\{2\}$.

in Sp_4: $\rightsquigarrow$ $y_I(a) = y_1(a)\, y_3(a)$

σ lifts to GL_4 $\quad y_1(a) \longleftrightarrow y_3(a)$, $y_2(a) \longleftrightarrow y_2(a)$ $\quad y_{II}(a) = y_2(a)$

Back to $A_3 \to B_2$ case

$$s_1 s_3 s_2 s_1 s_3 s_2 \rightsquigarrow s_{I} s_{II} s_{I} s_{II}$$

$$\| \qquad\qquad\qquad \|$$

$$s_2 s_1 s_3 s_2 s_1 s_3 \longrightarrow s_{II} s_{I} s_{II} s_{I}$$

$$y_1(a) y_3(a) y_2(b) y_1(c) y_3(c) y_2(d) = y_{I}(a) y_{II}(b) y_{I}(c) y_{II}(d)$$

$$\| \qquad\qquad\qquad \|$$

$$??? \qquad\qquad y_{II}(a') y_{I}(b') y_{II}(c') y_{I}(d')$$

$132\underline{132} \sim 1\underline{323}12 \sim 123\underline{212} \sim 12\underline{3}\underline{1}21 \sim \underline{121}321 \sim 21\underline{232}1$
$\sim 2132\underline{31} \sim 213213$

$$y_1(a) y_3(a) y_2(b) y_1(c) y_3(c) y_2(d) = y_1(a) \underline{y_3(a) y_2(b) y_3(c)} y_1(c) y_2(d)$$

$$= y_1(a) y_2\left(\frac{bc}{a+c}\right) y_3(a+c) \underline{y_2\left(\frac{ab}{a+c}\right) y_1(c) y_2(d)}$$

$$= y_1(a) y_2\left(\frac{bc}{a+c}\right) \underline{y_3(a+c) y_1\left(\frac{cd(a+c)}{A}\right)} y_2\left(\frac{A}{a+c}\right) y_1\left(\frac{abc}{A}\right) \quad \text{set } A = ab+ad+cd$$

$$= \underline{y_1(a) y_2\left(\frac{bc}{a+c}\right) y_1\left(\frac{cd(a+c)}{A}\right)} y_3(a+c) y_2\left(\frac{A}{a+c}\right) y_1\left(\frac{abc}{A}\right) \quad \text{set } B = aA + cd(a+c)$$

$$= y_2\left(\frac{bc^2d}{A}\right) y_1\left(\frac{B}{A}\right) \underline{y_2\left(\frac{abc/(a+c)}{B/A}\right) y_3(a+c) y_2\left(\frac{A}{a+c}\right)} y_1\left(\frac{abc}{A}\right)$$

$$= y_2\left(\frac{bc^2d}{A}\right) y_1\left(\frac{B}{A}\right) y_3\left(\frac{B}{A}\right) y_2(\;?\;) y_3\left(\frac{abc}{A}\right) y_1\left(\frac{abc}{A}\right)$$

Here $? = \frac{abc\,A}{(a+c)B} + \frac{A}{a+c} = \frac{A}{a+c}\left(\frac{abc}{B}+1\right) = \frac{A^2}{B}.$

So $y_{I}(a) y_{II}(b) y_{I}(c) y_{II}(d) = y_{II}(a') y_{I}(b') y_{II}(c') y_{I}(d')$

where $a' = \frac{bc^2d}{B}$, $b' = \frac{B}{A}$, $c' = \frac{A^2}{B}$, $d' = \frac{abc}{A}$

where $A = ab+ad+cd$, $B = aA + cd(a+c)$.

Here $\sigma(y_2(a_1) y_1(a_2) y_3(a_3)) = y_2(a_1) y_3(a_2) y_1(a_3)$

$$= y_2(a_1) y_1(a_3) y_3(a_2)$$

$(a,b,c) \longmapsto y_2(a) y_1(b) y_3(c)$

$\mathbb{R}^3_{>0} \xrightarrow{\cong} U^-_{s_2 s_1 s_3, >0}$ So $U^{-,\sigma}_{s_2 s_1 s_3, >0} = \{(a,b,b)\}$.

So far, we have the Coxeter relation for any rank 2 gps.

$\rightsquigarrow$ Construction of $U^-_{\geq 0}$ for any Kac-Moody gp.

Similarly, we have $U^+_{\geq 0}$.

The torus part. Consider $T(\mathbb{R}) \simeq \overset{\text{coweight lattice}}{Y} \otimes_{\mathbb{Z}} \mathbb{R}^\times \simeq (\mathbb{R}^\times)^{\operatorname{rank} G}$

Let $T_{>0}$ be the identity component of $T(\mathbb{R})$. Then $T_{>0} \simeq \mathbb{R}_{>0}^{\operatorname{rank}(G)}$.

Def. $G_{\geq 0} = \langle U^+_{\geq 0}, T_{>0}, U^-_{\geq 0} \rangle$ is the submonoid gen. by $T_{>0}$ and $x_i(a), y_i(a)$ for $i \in I$ and $a > 0$.

Generators: $t \in T_{>0}$, $x_i(a), y_i(a)$ for $i \in I$, and $a > 0$.

relations: $y_i(a) y_i(b) = y_i(a+b)$, $x_i(a) x_i(b) = x_i(a+b)$.

Cox. relation for y_i's, for x_i's.

$t x_i(a) = x_i(\alpha_i(t) a) t$, $t y_i(a) = y_i(\alpha_i(t)^{-1} a) t$

$x_i(a) y_j(b) = y_j(b)(x_i(a))$ for $i \neq j$

$x_i(a) y_i(b) = y_i\left(\frac{b}{1+ab}\right) \alpha_i^\vee(1+ab) x_i\left(\frac{a}{1+ab}\right)$ $\forall i \in I, a, b > 0$.

Decomposition into cells

$$U^\pm_{\geq 0} = \coprod_{w \in W} U^\pm_{w,>0}, \text{ and } U^\pm_{w,>0} \simeq \mathbb{R}_{>0}^{\ell(w)}$$

For $w_1, w_2 \in W$, set

$$G_{w_1,w_2,>0} = U^+_{w_1,>0} T_{>0} U^-_{w_2,>0} \simeq U^+_{w_1,>0} \times T_{>0} \times U^-_{w_2,>0} \simeq \mathbb{R}_{>0}^{\ell(w_1)+\ell(w_2)+\operatorname{rank}(G)}.$$

Thm. (1) $G_{\geq 0} = U^+_{\geq 0} T_{>0} U^-_{\geq 0} = U^-_{\geq 0} T_{>0} U^+_{\geq 0}$.

(2) $G_{\geq 0} = \coprod G_{w_1,w_2,>0}$ decomposition into cells.

Pf (1) follows easily from the relations.

(2) Since $U^+_{\geq 0} = \coprod_{w_1 \in W} U^+_{w_1,>0}$ $\quad U^-_{\geq 0} = \coprod_{w_2 \in W} U^-_{w_2,>0}$

So $U^+_{\geq 0} T_{>0} U^-_{\geq 0} = \coprod G_{w_1,w_2,>0}$

here the disjoint union follows from

$$\begin{tikzcd}
U^+ \times T \times U^- \arrow[r, "\cong"] & U^+ T U^- \subseteq G \\
\coprod U^+_{w_1,>0} \times T_{>0} \times \coprod U^-_{w_2,>0} \arrow[u, hook] \arrow[r] & \bigcup G_{w_1,w_2,>0} \arrow[u, hook]
\end{tikzcd}$$

this must be a disjoint union

$\square$

Change from $\mathbb{R}_{>0}$ to any semifield K.

Def. $U^{\pm}(K)$ is defined to be the (abstract) monoid with generators (i,a) for $i\in I$, $a\in K$, subject to relations: $(i,a)(i,a')=(i,a+a')$

Coxeter relation.

Key observation: the change-of-coordinate functions in the Coxeter relation are subtraction free, and thus can be defined over any semifield.

Rmk. We will use $y_i(a)$ for $(i,a)\in U^-(K)$

$x_i(a)$ for $(i,a)\in U^+(K)$.

Def. $G(K)$ is the (Kac-Moody) monoid gen. by $t\in \overline{K^{\mathrm{rank}(G)} \cong Y\otimes K}$,

$x_i(a), y_i(a)$ for $i\in I$, $a\in K$, relation similar to $G_{\geq 0}$.

Consider the case $K_1=\{1\}$ semifield of 1 elt.

$$U^-(K_1)=\coprod_{w\in W} U^-_w(K_1)$$

where $U^-_w(K_1)=\{y_1(a_1)\cdots y_n(a_n) ; a_i\in K_1\}$ is a ~~singleton~~ singleton for any reduced expression $w=s_1\cdots s_n$

In other words, $U^-(K) \cong W$ as sets

Note that $U^-(K_1)$ is gen. by i for $i\in I$

relation $i\cdot i=i$ $\quad (y_i(a)y_i(b)=y_i(a+b))$

Coxeter relation.

Thus $U^-(K_1)$ is the monoid associated to W.

$U^+(K_1)$ is the monoid associated to W

Thus $G(K_1)=\coprod_{w_1,w_2,>0} G_{w_1,w_2,>0} = W\times W$

product of the monoids.

Here $T(K_1)$ is a singleton.

Prop. $G_{w_1,w_2}(K)\, G_{w_1',w_2'}(K) \subseteq G_{w_1 * w_1', w_2 * w_2'}(K)$

Here $*$ is the monoid product on W.

Pf. We have the semifield homomorphism $K\to K_1$.

the relations on the Kac-Moody monoid is compatible with the base change $K\to K_1$. So

$G(K) \longrightarrow G(K_1)$ hom. of monoid

i.e.
$$\begin{tikzcd} G(k)\times G(k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & G(k) \arrow[d] \\ G(k_1)\times G(k_1) \arrow[r] & G(k_1) \end{tikzcd}$$

Now
$$\begin{tikzcd} G_{w_1,w_2}(K)\times G_{w_1',w_2'}(K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & G(k) \arrow[d] \\ (w_1,w_2)\times(w_1',w_2') \arrow[r, mapsto, "\text{monoid action}"'] & G(k_1)=W\times W \end{tikzcd}$$
$$\longmapsto (w_1 * w_1', w_2 * w_2')$$

So $G_{w_1,w_2}(K)\, G_{w_1',w_2'}(K) \subseteq G_{w_1*w_1', w_2*w_2'}(K)$ $\square$

Another application of semifield

Def. A map $K^n \to K$ is called admissible if it is of the form $f = f_1/f_2$ where $f_1, f_2 \in \mathbb{Q}_{\geq 0}[x_1,\cdots,x_n]$.

A map $K^n \to K^m$ is called admissible if $K^n \to K^m \to K$ is admissible for each coordinate.

A map $K^n \to K^n$ is called bi-admissible if it is admissible, bijective, and the inverse is admissible.

~~Prop~~ Recall that $U^-_{w,>0}$ is indep. of the choice of reduced expressions.

Then $w = s_1\cdots s_n = t_1 \cdots t_n$ two reduced expressions

$$\begin{tikzcd} (a_1,\cdots,a_n)\in \mathbb{R}^n_{>0} \arrow[r, dashed, "\tilde{\iota}_{\underline{w_1},\underline{w_2}}"] \arrow[d] \arrow[dr] & \mathbb{R}^n_{>0} \ni (b_1,\cdots,b_n) \arrow[d] \arrow[dl] \\ y_{s_1}(a_1)\cdots y_{s_n}(a_n) \in U^-_{w,>0} & \ni y_{t_1}(b_1)\cdots y_{t_n}(b_n) \end{tikzcd}$$

$\underline{w_1} = s_1 \cdots s_n$
$\underline{w_2} = t_1 \cdots t_n$
reduced expressions of w.

Prop. The bijective map $\mathbb{R}^n_{>0} \xrightarrow{\iota_{\underline{w_1},\underline{w_2}}} \mathbb{R}^n_{>0}$ is bi-admissible.

pf. Let K be the universal semifield of n variables. $(a_1,\cdots,a_n)$

Then
$$\begin{tikzcd} K^n \arrow[rr, "\text{bij.}"] \arrow[dr, "\text{bij.}"] \arrow[d, "\text{using } \underline{w_1}"'] & & K^n \arrow[dl, "\text{bij.}"'] \arrow[d, "\text{using } \underline{w_2}"] \\ \mathbb{R}^n_{>0} \arrow[rr, "\iota_{\underline{w_1},\underline{w_2}}"] \arrow[dr] & U^-_w(K) & \mathbb{R}^n_{>0} \arrow[dl] \\ & U^-_{w,>0} & \end{tikzcd}$$

$K \to \mathbb{R}_{>0}$

$(a_1,\cdots,a_n) \mapsto (f_1,\cdots,f_n)$

here $f_1,\cdots f_n \in K$ are admissible functions on $a_1,\cdots a_n$.

So $\iota_{\underline{w_1},\underline{w_2}}$ is admissible,

Similarly, the inverse is also admissible.

Prop.

$$\begin{tikzcd} U^+_{w_1,>0} \times T_{>0} \times U^-_{w_2,>0} \arrow[rr, squiggly, dashed, "\text{biadmissible}"] \arrow[dr, "\text{bij.}"'] & & U^-_{w_2,>0} \times T_{>0} \times U^+_{w_1,>0} \arrow[dl, "\text{bij.}"] \\ & G_{w_1,w_2,>0} & \end{tikzcd}$$

Rmk. $U^+_{w,>0} \longrightarrow ?$ is admissible if $\mathbb{R}^{\ell(w)}_{>0} \xrightarrow{i_{\underline{w}}} U^+_{w,>0} \longrightarrow ?$ is admissible.

By the previous prop, this def is indep. of the choice of reduced expressions of $\underline{w}$.