Semifield

Def. A <u>semifield</u> is a set k with $+$, $\cdot$ s.t.

- $(K,+)$ is commutative and associative (commutative semigp).
- $(K,\cdot)$ is an abelian group ← a field is not a semifield.
- distributive law.

Eg. $\mathbb{R}_{>0}$

$$\mathbb{R}(t)_{>0} := \left\{ t^e \frac{f_1(t)}{f_2(t)} \;\middle|\; \begin{array}{l} e\in\mathbb{Z},\ f_1(t), f_2(t)\in\mathbb{R}[t] \\ \text{s.t. } f_1(0), f_2(0)\in\mathbb{R}_{>0} \end{array} \right\}$$

Eg. Tropical $(\mathbb{Z}, \oplus, \odot)$.

$a\oplus b = \min\{a,b\}$

$a\odot b = a+b$

$\mathbb{R}(t)_{>0} \twoheadrightarrow \mathrm{Trop}\,\mathbb{Z}$ semifield homomorphism

$f \mapsto$ deg of lowest term

Remark. $\mathbb{R}(t)_{>0} = \{$ coeff of highest term $>0 \}$

$\mathrm{Trop}\,\mathbb{Z}'$: $a\oplus' b = \max\{a,b\}$

Then $\mathbb{R}(t)'_{>0} \twoheadrightarrow \mathrm{Trop}\,\mathbb{Z}'$

Remark. $\mathbb{R}\{\{t\}\} = \bigcup_{n\in\mathbb{N}} \mathbb{R}(t^{1/n})$.

Puiseux series

Define $\mathbb{R}\{\{t\}\}_{>0} = \bigcup_{n\in\mathbb{N}} \mathbb{R}(t^{1/n})_{>0}$

Then

$\mathbb{R}\{\{t\}\}_{>0} \twoheadrightarrow \mathrm{Trop}\,\mathbb{Q}$

eg. $\mathbb{k}_1 = \{1\}$. $1+1=1$. $1\cdot 1 = 1$.

Any semifield k have $k\to k_1$ semifield homo

Upshot: $G(k) \xrightarrow[\text{base change}]{} G(k')$ ← skeleton of $G(k)$

Week 2

Last time: $GL_{n,\geq 0}$ submonoid of $GL_n(\mathbb{R})$

gen. by $T_{>0}$, $x_i(a), y_i(a)$. $1\le i\le n-1$, $a>0$.

Big picture

$GL_n(\mathbb{R}_{>0}) \rightsquigarrow$ Replace G by any reductive gp or Kac Moody gp.

Replace $\mathbb{R}_{>0}$ by any semifield $\rightsquigarrow G(k)$

Eg of semifield.

(i). $\mathbb{R}_{>0}$.

(ii) $\mathrm{Trop}\,\mathbb{Z}$

(iii) k_1

Universal semifield

Def. Let $\mathbb{Q}(z_1,\dots,z_m)$ be the field of rational functions with variables $z_1,\dots,z_m$.

Let $\mathbb{Q}(z_1,\dots,z_m)_{>0} \subseteq \mathbb{Q}(z_1,\dots,z_m)$ be minimal subsemifield containing $z_1,\dots,z_m$.

$\mathbb{Q}(z_1,\dots,z_m)_{>0}$ is called the <u>universal semifield</u>.

Rmk. Let $m=1$.

Consider $z^2-z+1 \in \mathbb{Q}(z)$.

$$z^2-z+1 = \frac{z^3+1}{z+1} \in \mathbb{Q}(z)_{>0}.$$

So the coefficients can be negative.

Lemma. (universal property)

Let K be a semifield, and $y_1,\dots,y_m\in K$.

Then $\exists!$ semifield homo

$$\mathbb{Q}(z_1,\dots,z_m)_{>0} \to K$$
$$z_i \mapsto y_i$$

Rmk. This homo may not be surj. but good thing is we can choose $y_1,\dots,y_m$ freely. That's usually enough in practice.

pf. (Uniqueness is obvious if well-defined.

i.e., if P_1, Q_1, P_1, Q_2 are polynomials with m variables and non-negative coefficients.

and $\dfrac{P_1}{Q_1} = \dfrac{P_2}{Q_2}$ then

$$\frac{P_1(y_1,\cdots,y_m)}{Q_1(y_1,\cdots,y_m)} = \frac{P_2(y_1,\cdots,y_m)}{Q_2(y_1,\cdots,y_m)}$$

This is obvious since

$$\frac{P_1}{Q_1} = \frac{P_2}{Q_2} \Rightarrow P_1Q_2 = P_2Q_1$$

$$\Rightarrow \underset{\substack{\| \\ P_1(\underline{y})\,Q_2(\underline{y}) \\ \uparrow \\ K}}{P_1Q_2(\underline{y})} = \underset{\substack{\| \\ P_2(\underline{y})\,Q_1(\underline{y}) \\ \| \\ K}}{P_2Q_1(\underline{y})}$$

$$\Rightarrow \frac{P_1(\underline{y})}{Q_1(\underline{y})} = \frac{P_2(\underline{y})}{Q_2(\underline{y})} \qquad \square$$

Upshot. Later we will use

$X(k)$ ← misterious semifield.
↑ a variety

$$\begin{array}{ccc} X(\mathbb{Q}(z_1,\cdots,z_n)_{>0}) & \xrightarrow{\text{base change}} & X(k) \\ \big\downarrow \text{def ``Positive structure''} & & \\ X(\mathbb{Q}(z_1,\cdots,z_n)) & & \end{array}$$

$X(\mathbb{Q}(z_1,\cdots,z_n))$ ← something we understand more

variety over field.

Lie theory 101

(Weyl gps, Bruhat decompositions, Hecke algebras).

Def. Let V be f.d. Euclidean space (inner product space$/\mathbb{R}$). and $\langle\, ,\, \rangle$ be the inner product.

Let $\alpha \in V$. nonzero.

$S_\alpha : V \to V$ reflection along α.

$$v \mapsto v - \frac{2(\alpha, v)}{\langle \alpha, \alpha\rangle}\alpha.$$

A root system is a non-empty, finite subset Φ of V s.t.

(i) $\forall \alpha \in \Phi$, $S_\alpha \Phi = \Phi$ ← no need span V.

(ii) $\forall \alpha, \beta \in \Phi$, $\dfrac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}$

(A root system is reduced if $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$).

Let $V = \mathbb{R}^n$. $\{e_1,\cdots,e_n\}$ is a orthonormal basis.

i.e., $\langle e_i, e_j\rangle = \delta_{ij}$

Eg. (Type A_{n-1}, $G = GL_n$ or SL_n)

$\Phi = \{e_i - e_j,\ i \neq j\}$ $\quad \Sigma = \{e_1 - e_2, \cdots, e_{n-1} - e_n\}$.

Eg. (Type B_n. $G = SO_{2n+1}$)

$\Phi = \{\pm e_i \pm e_j : i \neq j\} \amalg \{\pm e_i, i\}$

$\Sigma = \{e_1 - e_2, \cdots, e_{n-1} - e_n, e_n\}$

Eg. (Type C_n, $G = Sp_{2n}$)

$\Phi = \{\pm e_i \pm e_j : i \neq j\} \amalg \{\pm 2e_i, i\}$.

$\Sigma = \{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\}$

Eg. (Type D_n, $G = SO_{2n}$)

$\Phi = \{\pm e_i \pm e_j,\ i \neq j\}$ $\quad \Sigma = \{e_1 - e_2, \cdots, e_{n-1} - e_n, \underline{\underline{e_{n-1} + e_n}}\}$

Rmk. Here we DO allow non-reduced root system, i.e., $\alpha, 2\alpha \in \Phi$ is allowed.

Eg. (Type BC_n).

$\Phi = \{\pm e_i \pm e_j\} \amalg \{\pm e_i\} \amalg \{\pm 2e_i\}$

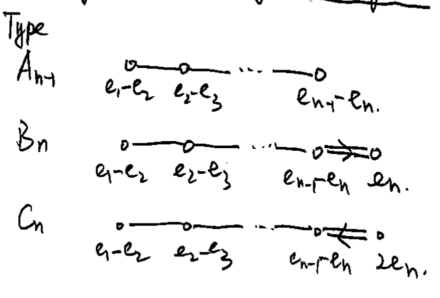
occurs in the ramified unitary gp over p-adic fields. (not used in this course)

Def. Choose a generic $\rho_0 \in V$, s.t. $\langle \alpha, \rho_0\rangle \neq 0$ $\forall \alpha \in \Phi$.
The positive roots $\Phi^+ := \{\alpha \in \Phi \mid \langle \alpha, \rho_0\rangle > 0\}$.

Def. The set of simple roots $\Sigma \subseteq \Phi^+$

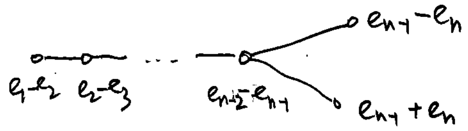
A positive root is simple if it is not the sum of two positive roots.

We may associate Dynkin diagrams



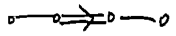
single edge: same length
edge: $\langle\, ,\, \rangle \neq 0$.
$\Rightarrow$: longer to shorter
no edge: 2
single edge: 3
double: 4 $\quad \theta = \frac{\pi}{2,3,4,6}$
triple: 6

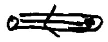
Type D_n



Type A–D are called <u>classical</u> types.

$E_{6,7,8}$

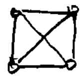
F_4: 

G_2: 

Type E, F, G are called <u>exceptional</u> types.

Affine types: $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}, \tilde{G}$.

  Loop gps or p-adic gps

None-affine types. 

Notation.

  Φ root system

  Σ simple roots

  $S = \{ s_\alpha : \alpha \in \Sigma \}$ simple reflections.

Def. Let $W \subseteq GL(V)$ be the subgp gen. by S.

  This is the <u>Weyl</u> group of (Φ, Σ).

Eg. Type A_{n-1}. $G = GL_n$. $W = S_n$.

  Type B_n/C_n, $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$

  Type D_n $W = S_n \ltimes \left((\mathbb{Z}/2\mathbb{Z})^n / (\mathbb{Z}/2\mathbb{Z}) \right)$ even parity

  Type G_2 $W = D_{12}$ dihedral gp of order 12.

  Type E/F more complicated.

Def. <u>Coxeter group</u> (W, S).

  W gp.

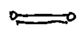
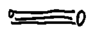
  S set of generators, s.t. relations

  - $s^2 = 1$ $\forall s \in S$
  - $(st)^{m_{st}} = 1$ $\forall s \neq t \in S$.

  $m_{st} = m_{ts} \in \{2, 3, \dots, \infty\}$.

If $m_{st} = m_{ts} \in \{2, 3, 4, 6, \infty\}$, (W, S) is called <u>crystallographic</u>. This is the Weyl gp of some Kac-Moody gp.

Eg. Rank 2 case.

$A_1 \times A_1$		$m=2$.	$st = ts$.
A_2		$m=3$.	$s=(1\,2)$ $t=(2\,3)$. st is 3-cycle.
B_2/C_2		$m=4$.	
G_2		$m=6$.	
$\tilde{A}_1$		$m=\infty$	

Rmk. $(st)^{m_{st}} = 1$ is equi. to $\underbrace{sts\cdots}_{m_{st}\text{-terms}} = \underbrace{tst\cdots}_{m_{st}\text{-terms}}$.

  <u>Coxeter relation</u>.

Fact. This is the longest element in the rank 2 coxeter group.

※ We assume the root system to be <u>reduced</u> now.

Thm. Let W be a Weyl gp. Then (W, S) is a coxeter gp.

Def. (Length function).

(i) Let W be a gp generated by a set of reflections S (elements of order 2).

  Let $w \in W$, <u>$\ell(w)$</u> is the minimum k s.t.

  $w = s_{i_1} \cdots s_{i_k}$ for $s_{i_1}, \dots, s_{i_k} \in S$.

  <u>reduced expression</u>

(ii). For Weyl gp W.

  Set $\ell'(w) = \left| \{ \alpha \in \Phi^+ ; w(\alpha) \in \Phi^- \} \right|$.

Fact. Let $w \in W$. $s = s_\alpha$ for $\alpha \in \Sigma$. Then

$$\ell'(ws) = \begin{cases} \ell'(w)+1 & \text{if } w(\alpha) \in \Phi^+ \\ \ell'(w)-1 & \text{if } w(\alpha) \in \Phi^- \end{cases} \qquad (*)$$

Exchange property: Let $w = s_1 \cdots s_k \in W$ and $s \in S$.

  If $\ell'(ws) < \ell'(w)$ then

  $s_1 \cdots s_k = s_1 \cdots \hat{s_i} \cdots s_k s$

pf. Let $s = s_\alpha$. Then $w(\alpha) \in \Phi^-$ by last Fact.

Then $\exists i$ s.t. $s_{i+1} \cdots s_k(\alpha) \in \Phi^+$

but $s_i s_{i+1} \cdots s_k(\alpha) \in \Phi^-$

Write s_i for s_{α_i}. Then

$s_{i+1} s_{i+2} \cdots s_k(\alpha) = \alpha_i$. $\left(s_i(\Phi^+ \setminus \{\alpha_i\}) \overset{\text{Prop1.4 Humphreys}}{=} \Phi^+ \setminus \{\alpha_i\}, \text{ as } \Phi \text{ reduced}\right)$.

So

$(s_{i+1} \cdots s_k) s_\alpha (s_{i+1} \cdots s_k)^{-1} = s_{\alpha_i} = s_i$. $\left(t s_\alpha t^{-1} \overset{\text{Prop1.2 Humphreys}}{=} s_{t\alpha}, t \in O(V)\right)$

Then

$s_{i+1} \cdots s_k s = s_i s_{i+1} \cdots s_k$

So $s_1 \cdots s_k = (s_1 \cdots s_{i-1})(s_i \cdots s_k)$

$= (s_1 \cdots s_{i-1})(\hat{s}_i s_{i+1} \cdots s_k s)$ □

Prop. Let $w \in W$. Then $\ell(w) = \ell'(w)$.

proof. ① $\ell'(w) \le \ell(w)$

Let $w = s_1 \cdots s_k$ where $k = \ell(w)$.

By (※), $\ell'(w) \le k$.

② $\ell(w) \le \ell'(w)$.

Suppose that $w = s_1 \cdots s_k$ with $k = \ell(w) > \ell'(w)$ for contradiction.

Assume furthermore that w is the smallest counterexample.

Then, $\ell(s_1 \cdots s_{k-1}) = \ell'(s_1 \cdots s_{k-1})$. $\left(\Rightarrow \ell'(s_1 \cdots s_{k-1} s_k) \underset{\text{by Fact}}{=} \ell'(s_1 \cdots s_{k-1}) - 1\right)$

Now we are in situation of exchange property

$w = s_1 \cdots s_{k-1} s_k = s_1 \cdots \hat{s}_i \cdots s_{k-1} \hat{s}_k$.

Thus $\ell(w) \le k-2$ contradiction. □.

Cor. (i). If $w(\Phi^+) \subseteq \Phi^+$, then $w = 1$.

(ii). W is a finite gp.

pf. (i). $\ell(w) = \ell'(w) = 0 \Rightarrow w = 1$.

(ii). By (i), $W \hookrightarrow \mathrm{Map}(\Phi^+, \Phi)$

Alternatively, $W \hookrightarrow \mathrm{Sym}\,\Phi$ □.

Cor. Let $w = s_1 \cdots s_k$ be a reduced expression.

Then $\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$

$= \{\alpha_k, s_k(\alpha_{k-1}), s_k s_{k-1}(\alpha_{k-2}), \cdots\}$.

sketch pf. $\underbrace{s_{i+1} s_{i+2} \cdots s_k}_{w'}(\underbrace{s_k \cdots s_{i+1}(\alpha_i)}_{\alpha'}) = \alpha_i \in \Phi^+$, but $s_i w' \alpha' = -\alpha_i \in \Phi^-$. □.

Thm. Let W be a weyl gp. Then (W, S) is a coxeter gp.

pf. We have $s^2 = 1$ $\forall s \in S$.

and coxeter relation $\forall s \neq t$ in S.

Let W' be the associated coxeter gp.

Then $W' \longrightarrow W$, $t_i \longmapsto s_i$ is a gp homo.

It is surjective by def of W.

Interlude: Lemma: Let $\alpha \in \Phi$. then $\exists\, w \in W$ s.t. $w(\alpha) \in \Sigma$ (but not related).

To prove this is injective, denote W' generated by $t_1 \cdots t_n$.

Let $t_1, \cdots, t_m \in$ kernel, with m minimal, i.e., $s_1 \cdots s_m = 1$ in W.

$\ell'(s_1 \cdots s_m) \equiv m \mod 2$. by exchange property.

So $2 \mid m$.

So $m = 2\ell$, we have

$s_1 \cdots s_\ell = s_1' \cdots s_\ell'$

As ℓ is minimum. Then

$s_1 \cdots s_\ell = s_1 \cdots \hat{s}_i \cdots s_\ell s_\ell'$

$t_1 \cdots t_\ell = t_1 \cdots \hat{t}_i \cdots t_\ell t_\ell'$

$\Rightarrow t_i \cdots t_\ell = t_{i+1} \cdots t_\ell t_\ell'$

As ℓ is minimum. $i = 1$.

$\Rightarrow t_1 \cdots t_\ell = t_2 \cdots t_\ell t_\ell'$

$= t_3 \cdots t_\ell t_\ell' t_\ell$

$= \cdots$

$= \underbrace{t_\ell t_\ell' t_\ell t_\ell' \cdots}_{\text{rank } 2}$

Similarly for LHS.

This must be coxeter relation.

So $t_1 \cdots t_m = 1$. □.

Thm. Let W be a gp gen. by finite set S of elements of order 2.

then (W,S) is a coxeter group iff exchange property holds for W.

Ref. Bump, Hecke algebra, 2010

Humphreys, Reflection groups and coxeter groups, 1990.