Goals: - TP cells in $\mathcal{B}$ satisfy the product structure

- TP cells in $\mathcal{B}$ has the Marsh-Rietsch parametrization

Here, TP cells are $\mathcal{B}_{v,w>0} := \mathcal{B}_{\geq 0} \cap \mathring{\mathcal{B}}_{v,w} = \overline{U^-_{\geq 0} \cdot B^+} \cap \mathring{\mathcal{B}}_{v,w}$

Ref: - Lustig, total positivity in partial flag varieties, 1998

- Rietsch, An algebraic cell decomposition ..., 1999
- Marsh-Rietsch, parametrization of flag varieties 2004
- Rietsch, closure relations for totally non-negative cells in G/P, 2006

- Bao-H, product structure and regularity theorem, 2022

General setting. $v \leq r \leq w$ $\quad Y \subset \mathring{\mathcal{B}}_{v,w}(\mathbb{R})$ conn. component

key assumption: $Y \subset r U^- \cdot B^+$

conclusion: $\bar{Y} \cap \mathring{\mathcal{B}}_{v,r} = C_{r,+}(Y)$ is a conn. comp of $\mathring{\mathcal{B}}_{v,r}(\mathbb{R})$

$\bar{Y} \cap \mathring{\mathcal{B}}_{r,w} = C_{r,-}(Y)$ is a conn. comp of $\mathring{\mathcal{B}}_{r,w}(\mathbb{R})$

$$\begin{tikzcd}
(C_{r,+}, C_{r,-}) : Y \arrow[d, hook] & \simeq & (\bar{Y} \cap \mathring{\mathcal{B}}_{v,r}) \times (\bar{Y} \cap \mathring{\mathcal{B}}_{r,w}) \arrow[d, hook] \\
\bar{Y} \cap r U^- B^+ & \subset & (\bar{Y} \cap \mathring{\mathcal{B}}_r) \times (\bar{Y} \cap \mathring{\mathcal{B}}^r)
\end{tikzcd}$$

Pf: Last time, we showed $\dot{r} B^+ \in \bar{Y}$

Here, we use for dominant coweight λ, $\lim_{t \to \infty} \lambda(t) u \lambda(t)^{-1} = 1 \quad \forall u \in U^-$

We have $C(C_{r,+}, C_{r,-})$:

$$\begin{array}{ccc}
r u^- \cdot B^+ & \xrightarrow{\simeq} & \mathring{B}_r \times \mathring{B}^r \\
\cup\!\shortmid & & \cup\!\shortmid \\
\bar{Y} \cap r u^- \cdot B^+ & \longrightarrow & (\overline{C_{r,+}(Y)} \cap \mathring{B}_r) \times (\overline{C_{r,-}(Y)} \cap \mathring{B}^r) \\
\cup\!\shortmid & & \cup\!\shortmid \\
Y & \xrightarrow{\simeq} & C_{r,+}(Y) \times C_{r,-}(Y)
\end{array}$$

Consider the point $\dot{r} B^+ \in \bar{Y} \cap r u^- B^+$

But $C_{r,\pm}(\dot{r} \cdot B^+) = r \cdot B^+$

So $\dot{r} \cdot B^+ \in \overline{C_{r,+}(Y)}$ $\quad \dot{r} \cdot B^+ \in \overline{C_{r,-}(Y)}$

Now, we consider

$$\begin{array}{ccc}
\bar{Y} \cap r u^- \cdot B^+ & \simeq & (\overline{C_{r,+}(Y)} \cap \mathring{B}_r) \times (\overline{C_{r,-}(Y)} \cap \mathring{B}^r) \\
\cup\!\shortmid & & \\
\bar{Y} \cap \mathring{B}_{r,r} & \simeq & C_{r,+}(Y) \times \{\dot{r} \cdot B^+\}
\end{array}$$

So $C_{r,+}(Y) = \bar{Y} \cap \mathring{B}_{r,r}$

it is a union of conn. comp. of $\mathring{B}_{r,r}$

Similar for $C_{r,-}(Y)$

$\square$

In general,

$$\begin{tikzcd}
{[v,w]} \arrow[r, "\text{cut at } v'"] \arrow[dr, squiggly] & {[v,v'] \times [v',w]} \arrow[d, "\text{cut at } w'"] \\
& {[v',w'] \times [w',w]}
\end{tikzcd}$$

Thm. Fix $v \le w$, Y a connected component of $\mathring{B}_{v,w}(\mathbb{R})$

Define $Y_{v',w'} := \bar{Y} \cap \mathring{B}_{v',w'}(\mathbb{R})$ $\forall\ v \le v' \le w' \le w$

key assumption $Y_{v',w'} \subseteq r u^{-} \cdot B^{+}$ $\forall\ v' \le r \le w$

Then - $Y_{v',w'} \simeq Y_{v',r} \times Y_{r,w'}$ (product structure)

Consequences: (1) $Y_{v',w'}$ is a conn. comp of $\mathring{B}_{v',w'}(\mathbb{R})$

(2) $\overline{Y_{v',w'}} = \bigsqcup_{v' \le v'' \le w'' \le w'} Y_{v'',w''}$

(3) $Y_{v',w'} \simeq (\mathbb{R}_{>0})^{\ell(w')-\ell(v')}$

Remark. The key point of the product structure is that C_r gives the isomorphism not only for a single $Y_{-,-}$, but for the union of $Y_{\le r, \ge r}$ as well.

So it reflects the geometric/topological properties on various $Y_{-,-}$.

Pf of the consequences:

① $Y_{v,w} \xrightarrow{C_{v',+}} Y_{v',w} \xrightarrow{C_{w',-}} Y_{v',w'}$

$\uparrow$ conn. comp of $\mathring{B}_{v,w}$ $\Rightarrow$ — of $\mathring{B}_{v',w}$ $\Rightarrow$ — of $\mathring{B}_{v',w'}$

② $\subseteq$: since $\overline{\mathring{B}_{v',w'}(\mathbb{R})} = \coprod_{v' \le v'' \le w'' \le w'} \mathring{B}_{v'',w''}(\mathbb{R})$

so $\overline{Y_{v',w'}} \subseteq \bar{Y} \cap \overline{\mathring{B}_{v',w'}(\mathbb{R})} = \coprod \bar{Y} \cap B_{v'',w''}(\mathbb{R})$

$= \coprod Y_{v'',w''}$

On the other hand, for $v' \le v'' \le w'' \le w'$,

$Y_{v'',w''} = C_{w'',-}(Y_{v'',w'}) \subseteq \overline{Y_{v'',w'}}$

$Y_{v'',w'} = C_{v'',-}(Y_{v',w'}) \subseteq \overline{Y_{v',w'}}$

③ Facts from Weyl groups:

Given $v' \le w'$, $\exists$ a sequence

$v' = v_0 <\cdot v_1 <\cdot \cdots <\cdot v_n = w'$ ($\le$ = Bruhat order, $<\cdot$ the corresponding covering relation. i.e. $x <\cdot y \iff x \le y$ and $l(y) = l(x)+1$)

Then product structure $\Rightarrow Y_{v',w'} \simeq Y_{v_0,v_1} \times \cdots \times Y_{v_{n-1},v_n}$

But $Y_{v_i,v_{i+1}}$ is a conn. component of $\mathring{B}_{v_i,v_{i+1}}(\mathbb{R}) \simeq \mathbb{R}^{\times}$

So $Y_{v_i,v_{i+1}} \simeq \mathbb{R}_{>0}$

Remark (Guess on the key assumption)

which conn comp of $\mathring{B}_{v,w}(\mathbb{R})$ satisfy the key assumption $\overset{?}{\Longleftrightarrow}$ choice of the sign of the pinnings on G

$\Big\updownarrow$ related

how to characterizes the total positive cells?

$\left(\overset{?}{\Longleftrightarrow}\right.$ conn component of $\mathring{B}_{v,w}(\mathbb{R})$ + topological cell $\mathbb{R}_{>0}^{\ell(w)-\ell(v)}$

What are the conn. comp. of $\mathring{B}_{v,w}(\mathbb{R})$?

For the big cell, i.e. $\mathring{B}_{1,w_0}(\mathbb{R}) = B^+ w B^+ \cap B^- \cdot B^+(\mathbb{R})$

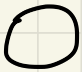
this is Arnold problem 1987-23(?), solved by Shapiro, Rietsch etc.

e.g. $(SL_2, T, B^{\pm}, x, y)$ $\quad$ $x : \mathbb{R} \to U^+$

$y : \mathbb{R} \to U^-$

Set $x' : \mathbb{R} \to U^+ \quad x'(a) := x(-a)$

$y' : \mathbb{R} \to U^- \quad y'(a) := y(-a)$

so $B =$  $\mathbb{RP}^1$

$\mathring{B}_{1,s}(\mathbb{R}) =$  $\mathbb{R}^{\times}$

Apply the general setting to the theory of total positivity

$$B_{\geq 0} = \overline{U^-_{\geq 0} \cdot B^+} .$$ Also $U^-_{\geq 0} = \coprod_{w \in W} U^-_{w,>0}$ is closed in U^-

<u>Step 1</u>: $B_{1,w>0} = U^-_{w,>0} \cdot B^+$ and is a conn. comp of $\mathring{B}_{1,w}(\mathbb{R})$

(This will be our Y)

pf: $U^-_{\geq 0}$ is closed in U^-, so $U^-_{\geq 0} \cdot B^+$ is closed in $U^- \cdot B^+$

$$\begin{aligned} B_{\geq 0} \cap \mathring{B}_{1,w} &= \overline{U^-_{\geq 0} \cdot B^+} \cap U^- \cdot B^+ \cap B^+ w \cdot B^+ \\ &= U^-_{\geq 0} \cdot B^+ \cap B^+ w \cdot B^+ = U^-_{w,>0} \cdot B^+ \end{aligned}$$

In particular, $U^-_{w,>0} B^+$ is closed in $\mathring{B}_{1,w}(\mathbb{R})$

Now

$$\begin{tikzcd} \mathbb{R}^{l(w)}_{>0} \arrow[r, "\sim"] & U^-_{w,>0} \arrow[r, hook] & U^-_{w,>0} \cdot B^+ \arrow[r] \arrow[d] & B^+ w \cdot B^+(\mathbb{R}) \simeq \mathbb{R}^{l(w)} \\ (a_1, a_2, \cdots) \arrow[r, mapsto] & y_{i_1}(a_1)\, y_{i_2}(a_2) & \mathring{B}_{1,w}(\mathbb{R}) \arrow[ur] & \end{tikzcd}$$

Use Brouwer's thm of invariance of domain:

$U = \mathbb{R}_{>0}^{\ell(w)} \subseteq \mathbb{R}^{\ell(w)}$ open, $f : U \to \mathbb{R}^{\ell(w)}$ is injective continuous

then $f(U)$ is open and $f : U \to f(U)$ is a homeomorphism

So in particular, $U^-_{w,>0} \cdot B^+$ is open in $\mathring{B}_{1,w}(\mathbb{R})$

So $U^-_{w,>0} \cdot B^+$ is a conn. comp. of $\mathring{B}_{1,w}(\mathbb{R})$ $\square$

<u>Step 2</u>: If $r \leq w$, then $U^-_{w,>0} \subseteq \dot{r} U^- B^+$.

Calculate $\dot{r}^{-1} U^-_{w,>0}$ directly

$w = s_{i_1} \cdots s_{i_n}$ reduced expression

$\dot{r}^{-1} y_{i_1}(a_1) \cdots y_{i_n}(a_n)$

Case 1: $r^{-1} s_{i_1} > r^{-1}$, then $\dot{r} y_{i_1}(a_1) \in U^- \dot{r}^{-1}$

So set $w_1 = s_{i_2} \cdots s_{i_n}$ induction on (w_1, r)
$r \leq w_1$

Case 2 $r^{-1} s_{i_1} < r^{-1}$, then $\dot{r} y_{i_1}(a_1) \dot{r} \in U^+$

Set $r' = s_{i_1} r < r$

$\dot{s}_{i_1}^{-1} y_{i_1}(a) = \alpha_{i_1}^\vee(a^{-1}) y_{i_1}(-a) x_{i_1}(a^{-1})$ (sl_2-calculation)

Now $\dot{r}^{-1} y_{i_1}(a_1) \cdots y_{i_n}(a_n)$

$\in \dot{r}^{-1} \check{\alpha}_{i_1}(>0) y_{i_1}(<0) x_{i_1}(>0) y_{i_2}(>0) \cdots y_{i_n}(>0)$

$\subseteq \dot{r}^{-1} y_{i_1}(<0) y_{i_2}(>0) \cdots y_{i_n}(>0) B^+$

$\subseteq \dot{u} \dot{r}'^{-1} y_{i_2}(>0) \cdots y_{i_n}(>0) B^+$

then induction on (w', v')

As a consequence, $B_{r,w>0} = C_{v,+}(B_{1,w,>0})$ is a conn. comp. of $\mathring{B}_{r,w}(\mathbb{R})$.

<u>Step 3</u> $B_{v,w,>0} = G_{\underline{v+}, \underline{w}, >0} \cdot B^+$ (Marsh-Rietsch parametrization)

positive subexpression of v in the fixed reduced expression $\underline{w}$ of w

$G_{\underline{v+}, \underline{w}}$ is a cartesian product of $y_i(>0)$ and $\dot{s}_i$

(i) parametrization $\subseteq B_{v,w,>0}$

First $G_{\underline{v+}, \underline{w}, >0} B^+ \subseteq$ top Deodhar comp. of $\mathring{B}_{v,w}(\mathbb{R})$

It remains to show that it is contained in $B_{\geq 0}$

$B_{\geq 0}$ is preserved by the left action of $y_i(>0)$ since it is the closure of $U^-_{\geq 0} \cdot B^+$

For the $\dot{s}_i$-term involved, one use the property of the positive subexpression

i.e. if the first term in $\underline{v_+}$ is s_i, then

$$\mathring{B}_{v,w}(\mathbb{R}) = \dot{s}_i \mathring{B}_{s_i v, s_i w}(\mathbb{R})$$

both length decrease by 1

By induction / Cr-map,

$$\mathring{B}_{v,w,>0} = \dot{s}_i \mathring{B}_{s_i v, s_i w >0}$$

$$\cup\!\!\!\mid$$

$$G_{\underline{v_+}, w, >0} \cdot B^+ = \dot{s}_i G_{s_i \underline{v_+}, \underline{s_i w}} \cdot B^+$$

ii) $B_{v,w,>0} \subseteq$ parametrization

Note that $G_{\underline{v_+}, \underline{w}, >0} \cdot B^+$ is a conn. comp. of $\mathring{B}_{\underline{v_+}, \underline{w}}(\mathbb{R})$ (the Deodhar comp)

& $B_{v,w,>0}$ is a conn. comp. of $\mathring{B}_{v,w}(\mathbb{R})$ (Open Richardson var.)

It remains to show that $B_{v,w,>0}$ does not intersects lower Deodhar component. (next time)

<u>Step</u> 4 For $v \le r \le w$ $\mathcal{B}_{v,w,>0} \subseteq \dot{r}\, U^- \cdot B^+$

By step 3, need to show that

$$\dot{r}^{-1}\, G_{\underline{v_+},\, \underline{w},\, >0} \subseteq U^- \cdot B^+$$

It is proved in a similar, but slightly more difficult way as in step 2. (Bao-H, lemma 5.1)