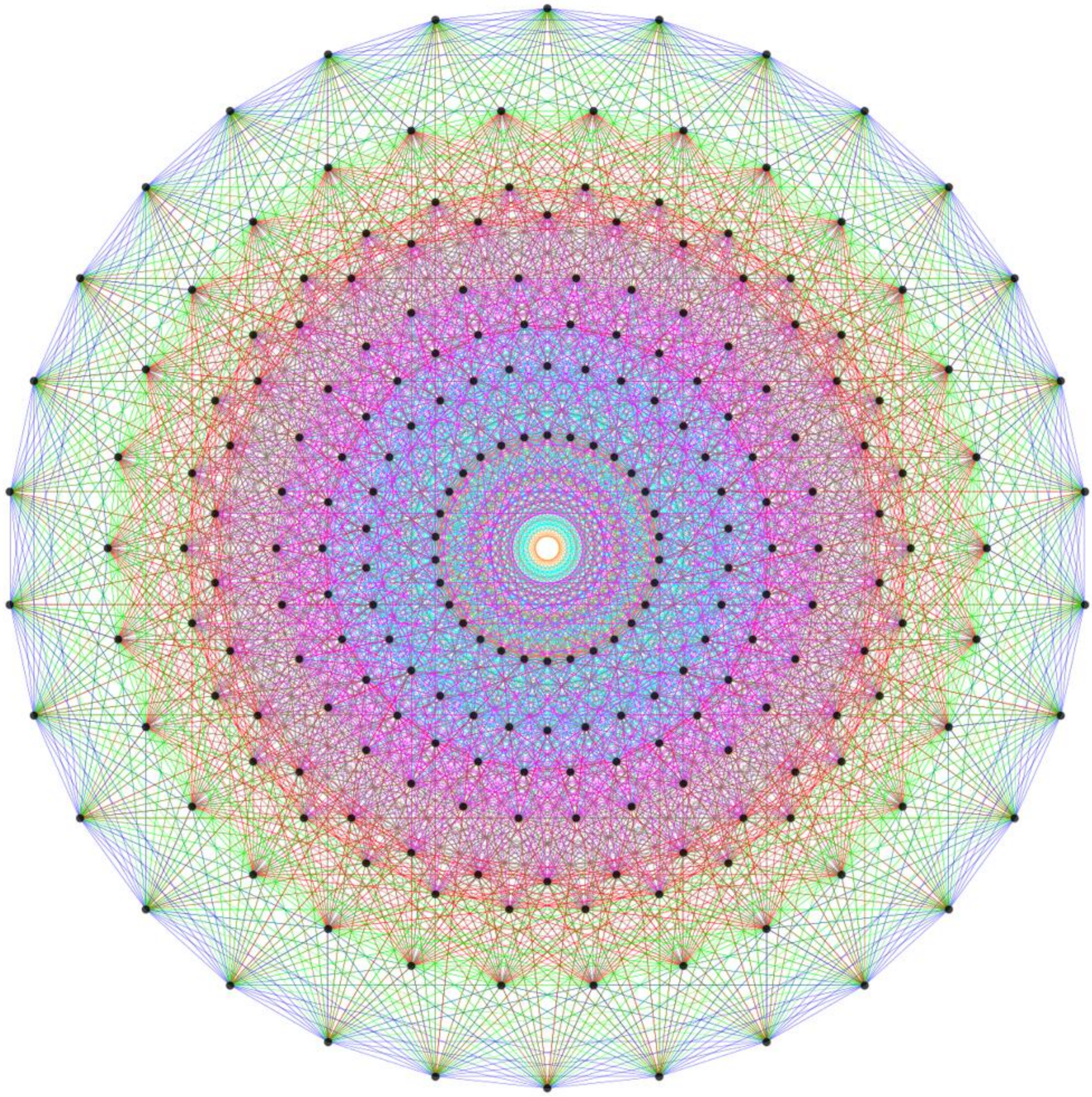

MATH6032 - Topics in Algebra II - 2021/22

Total positivity - Lecture 10

Lecturer: Prof.Xuhua He

Institute: The Chinese University of Hong Kong

Date: 2022/3/23

Previously, we consider $K = \mathbb{R}_{>0}$, a semifield. We have

Lemma 0.1

Let $K = \mathbb{R}_{>0}$. For any $w_1, w_2 \in W$ with $\operatorname{supp}(w_1) = \operatorname{supp}(w_2) = I$, and $g \in U^+_{w_1}(K)T(K)U^-_{w_2}(K)$, $\exists u_1 \in U^-_{w_0}(K), u_2 \in U^+_{w_1}(K), t \in T(K)$, s.t. , $gu_1 = u_1u_2t$. ♡

Goal of today's lecture: this is true when $K = \operatorname{Trop}\mathbb{Q}$ $(a \oplus b = \min(a,b), a \odot b = a + b)$.

✍ **Exercise 0.1** *Write the statement in lemma 0.1 explicitly for $GL_3(\operatorname{Trop}\mathbb{Q})$ and prove it directly.*

Today:Mainly mathematical logic.

Reference:Tarski's principle and the elimination of quantifiers by Richard G.Swan[1].

Then we show that lemma 0.1 holds when $K = k_{>0}$ for any real closed field k. In particular, it holds for real Puiseux series. Applying base change, it then holds for $K = \operatorname{Trop}\mathbb{Q}$.

Let $f \in \mathbb{Z}[x_1, \cdots, x_n]$. We will regard f as a function over a field F.

Definition 0.1

*An **atomic predicate** is of the form $f = 0$ for field F or $f = 0, f > 0$ for ordered field F.* ♣

We have the **logical connections**.

1. disjunction $P \vee Q$(P or Q);
2. conjunction $P \wedge Q$(P and Q);
3. negation $P \neq Q$(not P).

Example 0.1 *P is $(f = 0)$, then $\neg P$ is $(f \neq 0)$; P is $(f = 0)$,Q is $(g = 0)$, then $P \vee Q$ is $(fg = 0)$; $P \Rightarrow Q$ is equivalent to $(\neq P) \vee Q$.*

We have the **Quantifiers**:$\forall, \exists$

Definition 0.2

*The set of **elementary predicates** is the smallest class containing the atomic predicates and close under $\vee, \wedge, \neg$ and $\forall, \exists$. The set of **quantlfier - free elementary predicates** is the smallest class containing the atomic predicates and closed under $\vee, \wedge, \neg$ (and no quantifier is used).* ♣

Example 0.2 *Let F be a field. $\operatorname{char}(F) = p$ is equivalent to $p = 0$(quantifier-free) or $(\forall x)[px = 0]$(not quantifier-free) and both are elementary. $\operatorname{char}(F) = 0$ is equivalent to $\neg[p = 0]$ for all p, i.e. $(\neg[2 = 0]) \wedge (\neg[3 = 0]) \wedge \cdots$.*

The field we will consider here are algebraically closedfield and real closed field.

Definition 0.3

A field F is called real*(or **formally real**) if for any finite sequence $a_i \in F$, $\sum a_i^2 = 0$ if and only if $a_i = 0$. F is called **real closed** if it is real and no proper algebraic extention of F is real.* ♣

Fact *Any real field has a real closure (i.e. an algebraic extension that is real closed). This real closure is unique up to isomorphism.*

Lemma 0.2

Let F be a real field. Let $a \in F, a \neq 0$, then $F(\sqrt{a})$ is real iff $-a$ is not a sum of squares in F. ♡

Proof "$\Rightarrow$"If $-a = \sum c_i^2, a_i \in F$. Then in $F(\sqrt{a})$, we have $(\sqrt{a})^2 + \sum c_i^2 = 0$. Since $F(\sqrt{a})$ is real, $\sqrt{a} = 0 \Rightarrow a = 0$.

"$\Leftarrow$" If $F(\sqrt{a})$ is not real, then $\exists x_i, y_i \in F$, s.t. $\sum\left(x_i+y_i\sqrt{a}\right)^2=0$. So $\sum x_i^2+\sum y_i^2 a=0$ and $\sum x_i y_i \sqrt{a}=0$. Since (x_i, y_i) not all zero, y_i are not all 0. (otherwise $\sum x_i^2=0$, so $x_i=0$). Thus $-a=-\sum x_i^2 / \sum y_i^2=\left(\sum x_i^2\right)\left(\sum y_i^2\right) /\left(\sum y_i^2\right)^2$. ■

Definition 0.4

Two elementary predicates $P\left(x_1, \cdots, x_n\right)$ and $Q\left(x_1, \cdots, x_n\right)$ are equinalent in the theory of algebraically closed fields(resp.real closed field) if for any algebraically closed fields(resp.real closed field) F and $a_1, \cdots, a_n \in F, P\left(a_1, \cdots, a_n\right)$ is true iff $Q\left(x_1, \cdots, x_n\right)$ is true. ♣

Example 0.3 *$\left[x^2>9\right]$ and $\left[x^4>0\right]$ are equivalent in the theory of real closed field.*

Example 0.4 *Over $\mathbb{R}$, $[x \geq 0]$ is equivalent to $(\exists y)\left[x=y^2\right]$. Over $\mathbb{Q},[x \geqslant 0]$ is not equivalent to $(\exists y)\left(x=y^2\right)$.*

Principle 0.1 (Tarski principle)

An elementary predicate in the theory of algebraically closed field(resp. real) is equivalent to a quantifier - free elementary predicate. ♠

A simple observation is that let F, F' be an algebraically closed field(resp. real closed field), $F \subseteq F'$, then for any atomic predicate $P(f=0)$ or $P(f>0)$ and for $a_1, \cdots, a_n \in F$, $P(a_1, \cdots, a_n)$ is true in F iff it is true in F' (Here we may replace $f>0$ by $f=g^2, g \neq 0$).

A consequence of Tarski principle: Let S be an elementary statement/predicate in the theory of algebraically closed field(resp. real closed field). If it is true for one algebraically closed field(resp. real closed field), then it is true for all algebraically closed field(resp. real closed field).

Here is an application.

Theorem 0.1 (Hilbert's 17th problem, proved by Artin)

Let $f \in \mathbb{R}[x_1, \cdots, x_n]$, if $f(a_1, \cdots, a_n) \geq 0, \forall a_1, \cdots, a_n \in \mathbb{R}$, then f is a sum of squares in the quotient field $\mathbb{R}(x_1, \cdots, x_n)$. ♡

Proof Set $F=\mathbb{R}$, consider $F(\sqrt{-f})$. If f is not a sum of squares, then $F(\sqrt{-f})$ is real. Let K be a real closure of $F(\sqrt{-f})$. Then $-f=(\sqrt{f})^2$ in k. So $P:\left(\exists x_1\right)\left(\exists x_2\right) \cdots\left(\exists x_n\right)\left[f\left(x_1, \cdots x_n\right)<0\right]$ is true for K. By Tarski principle, P is true for $\mathbb{R}$. i.e. $\exists a_1, \cdots, a_n, f\left(a_1, \cdots, a_n\right)<0$. Contradiction. ■

Another application is the generalization of lemma 0.1

Proposition 0.1

Let k be a real closed field and $K=k_{>0}$. For any $w_1, w_2 \in W$ with $\operatorname{supp}(w_1)=\operatorname{supp}(w_2)=I$, and $g \in U_{w_1}^{+}(K) T(K) U_{w_2}^{-}(K), \exists u_1 \in U_{w_0}^{-}(K), u_2 \in U_{w_1}^{+}(K), t \in T(K)$, s.t. , $g u_1=u_1 u_2 t$. ♠

Proof Write $g u_1=u_1 u_2 t$ as elementary predicates. We identify $U_{w_0}^{-}(K) \simeq K^{l(w_0)}, U_{w_1}^{+}(K) \simeq K^{l(w_1)}, T(K) \simeq K^{\operatorname{rank}(G)}$. Then both sides are contained in $U_{w_0}^{-}(K) T(K) U_{w_1}^{+}(K) \simeq K^{l(w_0)+l(w_1)+\operatorname{rank}(G)}$. Under this identification

$$
\begin{gathered}
K^* \simeq U_{w_0}^{-}(K) \longrightarrow U_{w_0}^{-}(K) T(K) U_{w_1}^{+}(K) \simeq K^* \\
u_1 \longmapsto g u_1
\end{gathered}
$$

The map $K^* \to K^*$ involves the quotient of $\mathbb{Z}[x_1, \cdots]$. Similarly,

$$K^* \simeq U^-_{w_0}(K) \times U^+_{w_1}(K) \times T(K) \longrightarrow U^-_{\omega_0}(K)T(K)U^+_{w_1}(K) \simeq K^*$$

$$(u_1, u_2, t) \longmapsto u_1 t u_2$$

and the map involves the quotient of $\mathbb{Z}[x_1, \cdots]$.

Note that

$$f_1/f_2 = g_1/g_2 \Leftrightarrow [f_2 \neq 0] \wedge [g_2 \neq 0] \wedge [f_1 q_2 = f_2 g_1]$$

So $gu_1 = u_1 u_2 t$ is equivalent to $(\exists x_1) \cdots (\exists x_{l(w_0)+l(w_1)+\mathrm{rank}(G)})[P($elementary predicates from the coordinatewise equalitites $f_1/f_2 = g_1/g_2)]$.

We proved last week that the above statement is true for $\mathbb{R}$. So by Tarski principle, it is true for the real closed field k. So Since $[x_1 > 0] \wedge \ldots$, all the x_i are in $K = k_{>0}$. So we get a solution u_1, u_2, t over K. ■

Remark For the uniqueness, we may consider the statement $\neg\left((\exists x_i)\cdots\right) \wedge \left((\exists x_i')\cdots\right) \wedge \left([x_1 \neq x_1'] \vee [x_2 \neq x_2'] \vee \cdots\right))$.

Now we have the semifield homomorphism

$$\deg : K_1 = \mathbb{R}\{\{t\}\}_{>0} \to K_2 = \mathrm{Trop}\, \mathbb{Q}$$

$\mathbb{R}\{\{t\}\}_{>0}$ is the positive part of the real Puiseus series $\mathbb{R}\{\{t\}\}$ which is real closed.

For $g \in U^+_{w_1}(K_2)T(K_2)U^-_{w_2}(K_2)$, there exists $g' \in U^+_{w_1}(K_1)T(K_1)U^-_{w_2}(K_1)$ such that $\deg(g') = g$ since $K_1 \to K_2$ is surjective. By the theorem above, $\exists u_1' \in U_{w_0}(K_1), u_2' \in U^+_{w_1}(K_1), t' \in T(K_1)$ such that $g'u_1' = u_1' u_2' t'$. Let $u_1 = \deg(u_1'), u_2 = \deg(u_2'), t = \deg(t')$. Then $gu_1 = u_1 u_2 t$.

Remark Tarski principle shows the existence, but not the uniqueness. In fact, the uniqueness of (u_1, u_2, t) s.t. $gu_1 = u_1 u_2 t$ fails over $\mathrm{Trop}\, \mathbb{Q}$ (even for GL_2).

In the following we give a sketch of the prinf for Tarski principle for algebraically closed field. For real closed field, this is proved in a similar way, but more involved in §9[1].

1. It suffices to eliminate one quantifier at one time. Also $(\forall x)[P(x)]$ is equivalent to $\neg(\exists x)(\neg P(x))$ So we mainly consider the quentifier $\exists$. (not $\forall$.)
2. Any quantifier-free elementary predicate P is equivalent to $\forall(P_i)$, where $P_i = \wedge B_j$ and B_j is of the form A_j or $\neg A_j$ for atomic A_j.

   This is done using basic logic relations such as $(B_1 \vee B_2) \wedge (B_3 \vee B_4) = (B_1 \wedge B_3) \vee (B_1 \wedge B_4) \vee (B_2 \wedge B_3) \vee (B_2 \wedge B_4)$.

   Upshot: it suffices to consider the predicate of the form $(\exists x) \quad [f_1 = 0 \wedge f_2 = 0 \wedge \cdots \wedge g_1 \neq 0 \wedge g_2 \neq 0 \wedge \cdots]$. Note that f, g are polynomials in x, with coeffients as polynomials for other varlables. Thus in general, we can not reduce to monic polynomials.
3. Pseudo-monic form (w.r.t. x) is of the form $[c \neq 0] \wedge [Q(x)]$, where c is a polynomial not involving x, and is divisible by all the leading coefficients of all polynomials in $Q(x)$.

   Upshot: Any quantifier-free predicate is equivalent to $\vee P_i$, where P_i are quantifier-free pseudo-monic predicate.
4. Euclidean algorithm (and induction on degree of n)§7[1].
5. Over algebraically closed field, the pseudo monic form $(\exists x)[c \neq 0 \wedge g \neq 0]$ is equivalent to $c \neq 0$ if c does not involve x. And then to prove the pseudo-monic form $(\exists x)[c \neq 0 \wedge f = 0 \wedge g \neq 0] \Leftrightarrow$ a quantifier-free predicate.

Bibliography

[1] R Swan. *Tarski's Principle and the elimination of quantifiers*.