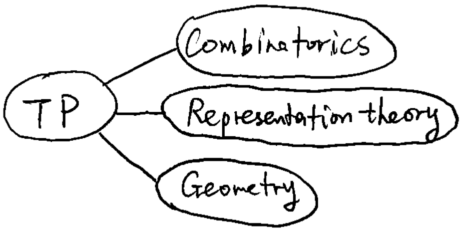
MATH 6032 Topics in Algebra 21-22
Introduction to Total Positivity

Week 1 Wed.



Applications:

- Cluster algebra (Fomin-Zelevinsky)
- higher Teichmüller theory (Fock-Goncharov)
- String theory, (amplituhedron (Arkani-Hamed).
- Tropical geometry
- Mirror symmetry
- Poset topology.

① Focus on Lie theory over semifield

② Focus on poset topology (Galashin-Karp-Lam, JAMS 2021).

References.

Lusztig. Total positivity in reductive groups 1994
part II 2019

series of papers after this.

Marsh-Rietsch Parametrization of TP flag 2004.

Bao-He Flags over semifield. 2021.

Postnikov Combinatorics of the Grassmanian

Speyer course note.

Lam CDM, 2014.

Ip course notes on cluster algebras.

__TP in reductive gps I__
Kac Moody in general

__Plan__

$GL_{n,>0} = GL_n(\mathbb{R}_{>0})$
(1930')

Replace $\mathbb{R}_{>0}$ by arbitrary semifield

Replace GL_n by arbitrary Kac-Moody gps

$G(k)$
(Lusztig II, 2019)

Def. (Schoenberg 1930, Gantmacher-Krein, 1935).
A matrix $g \in GL_n(\mathbb{R})$ is __totally positive__ (TP) (totally non-negative/TNN) if every minor is positive (non-negative).

Eg. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ g is TP. iff $a, b, c, d > 0$.
$\det g > 0$

Q: Find an explicit TP 3×3 matrix

Prop. There are $\binom{2n}{n} - 1$ minors in a $n\times n$ matrix.

Def (minor)
$I, J \subseteq \{1, 2, \dots, n\}$ with $|I| = |J|$, nonempty.
The __minor__ $\Delta_{I,J}(g) = \det(g_{ij})_{i\in I, j\in J}$.

Pf of prop.

$\#\{I, J \subseteq \{1, \dots, n\} \text{ with } |I| = |J|\}$

$\updownarrow$ bij (complement)

$\#\{I, J^c \subseteq \{1, \dots, n\} \text{ with } |I| + |J^c| = n\}$

$\updownarrow$ bij

$\#\{I \sqcup J^c \subseteq \{1, \dots, n, \bar{1}, \dots, \bar{n}\} \text{ with } |I \sqcup J^c| = n\} = \binom{2n}{n}$. □

Rmk. There are some redundancy in $\{\Delta_{I,J} > 0\}$.

Eg. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad a, b, c, \underset{\substack{\uparrow \\ \text{redundant}}}{d}, ad - bc > 0$

$$d = \frac{\det + bc}{a} > 0.$$

Def. (i) <u>gp</u> : A monoid with inverse.

(ii) <u>monoid</u> : A semigp with identity.

(iii). <u>Semigp</u> : A set with associative binary operation.

i.e., $(ab)c = a(bc)$.

(iv). $GL_n(\mathbb{R})$ gp

$GL_{n,>0}$ = set of TP matrices

$GL_{n,\geq 0}$ = set of TNN matrices.

Prop. $GL_{n,>0}$ is a sub-semigp. of $GL_n(\mathbb{R})$.
$GL_{n,\geq 0}$ is a sub-monoid

pf. Cauchy-Binet formula:

Let X be a $p \times q$ matrix. $(p \leq q)$.
Y be a $q \times p$ matrix.

$$\det(XY) = \sum_{I \in \binom{[q]}{p}} \det(X_{\underset{\substack{\uparrow \\ \text{all rows}}}{[p]}, I}) \det(Y_{I, \underset{\substack{\uparrow \\ \text{all columns}}}{[p]}}) \quad \square$$

Loewner-Whitney Thm.

$GL_{n,\geq 0}$ is the submonoid of $GL_n(\mathbb{R})$ generated by

$$T_{>0} = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} : a_i > 0 \right\}$$

$x_i(a)$, $y_i(a)$ $\quad a > 0$, $i = 1, \cdots, n-1$.

$$x_i(a) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & a & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \leftarrow (i, i+1) \text{ position.}$$

$$y_i(a) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & a & \ddots & \\ 0 & & & 1 \end{pmatrix} \leftarrow (i+1, i) \text{ position}$$

idea: For any $g \in GL_{n,\geq 0}$.

Find i, a s.t.

$x_i(-a) g$ is still TNN.

but has more zero entries than g.

until impossible.

do it for

$y_i(-a) g$ ...

Finally we get an element in $T_{>0}$. $\square$

Notation.

GL_n Lie gp $\quad GL_n \neq U_- T U^+$

$\mathfrak{gl}_n$ Lie alg. $\quad \mathfrak{gl}_n = \underset{\substack{\uparrow \\ \text{lower triangular}}}{\mathfrak{u}_-} \oplus \underset{\substack{\uparrow \\ \text{diagonal}}}{\mathfrak{h}} \oplus \underset{\substack{\uparrow \\ \text{upper triangular}}}{\mathfrak{u}_+}$

$$U^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$$

$$U^+ = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$$

$$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

Prop. (Cryer) $GL_{n,\geq 0} = U^-_{\geq 0} T_{>0} U^+_{\geq 0}$

$= U^+_{\geq 0} T_{>0} U^-_{\geq 0}$

$= \cdots$ can permute any two

$U^-_{\geq 0}$ is the submonoid gen. by $y_i(a)$, $i, a > 0$

$U^+_{\geq 0}$ is the submonoid gen. by $x_i(a)$, $i, a > 0$

Rmk. In GL_n, T normalizes $U^\pm$.

But $U^-, U^+ \neq U^+ U^-$.

pf. • $t x_i(a) = x_i(\alpha_i(t) a) t$ where

$\alpha_i(t) = t_i / t_{i+1}$ <u>root</u> $\qquad t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$

• $t y_i(a) = y_i(\alpha_i(t)^{-1} a) t$

• $x_i(a) y_j(b) = y_j(b) x_i(a) \quad i \neq j$

• $x_i(a) y_i(b) = y_i(\frac{b}{1+ab}) \alpha_i^\vee(1+ab) x_i(\frac{a}{1+ab})$

SL_2-calculation $\qquad$ if $1+ab \neq 0$

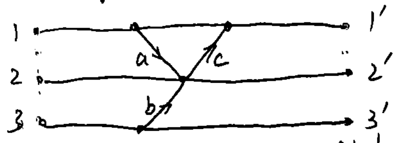
$$\alpha_i^\vee(c) := \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & & \\ & & & c^{-1} & \\ 0 & & & & \ddots \; 1 \end{pmatrix} \begin{matrix} \leftarrow i^{th} \text{ diagonal} \\ \leftarrow (i+1)^{th} \text{ diagonal} \end{matrix}$$

<u>coroot</u> $\square$

Combinatorial Model for $GL_{n,\geq 0}$

Directed acyclic planar network.

N:



source sink

no mark — weight = 1.

going left to right, weight of path = multiply edges.

$$M(N) = \begin{pmatrix} 1+ac & a & 0 \\ c & 1 & 0 \\ bc & b & 1 \end{pmatrix}$$

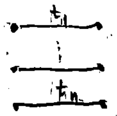
(i,j): all paths from source i to target j.

associated matrix.

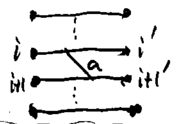
$$(\text{entry for } ij') = \sum_{\substack{\text{paths} \\ i \to j' \\ p}} \text{multiplication of weights of edges in } p$$

Thm. $g \in GL_n$ is TNN iff $g = M(N)$ for some network N with positive weights.

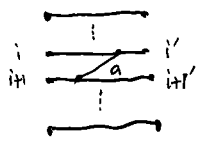
Pf. $(\Rightarrow)$ $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \longleftrightarrow$



$X_i(a) \longleftrightarrow$

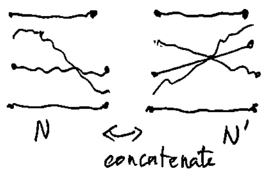


$y_i(a) \longleftrightarrow$



multiplication

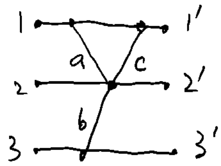
$$M(N)\, M(N') = M(N * N')$$



$(\Leftarrow)$. Lindström's Lemma.

$$\Delta_{I,J}(M(N)) = \sum_{\substack{\text{non-intersecting} \\ \text{paths } P: I \to J}} wt(P)$$

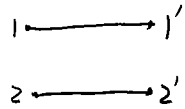
eg.



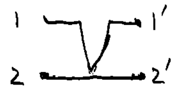
$$\begin{pmatrix} 1+ac & a & 0 \\ c & 1 & 0 \\ bc & b & 1 \end{pmatrix}$$

$$\Delta_{\{1,2\}\{1',2'\}} = (1+ac) - ac = 1.$$

paths:

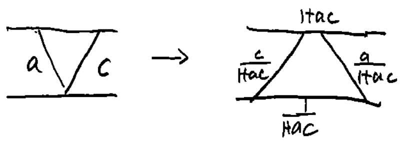


$\overset{wt}{\rightsquigarrow} 1$

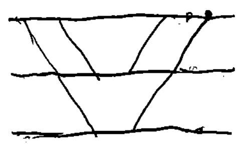


$\overset{wt}{\rightsquigarrow} ac$

$\overset{wt}{\rightsquigarrow} -ac$ as $1 \to 2'$, $2 \to 1'$



Rmk. Any network with positive weights is equivalent (having same $M(N)$) to the following network.



$GL_{n,\geq 0}$ geometric structure

Q: Nice decomposition into cells
Relations among cells.

A: Use Bruhat decomposition
answer is given by Weyl group elements.

$GL_n \longleftrightarrow S_n \longleftrightarrow GL_n$/semifield of 1 element

alg gp Weyl gp

a kind of "base change"

Semifield

Def. A <u>semifield</u> is a set k with $+, \cdot$ s.t.

- $(K,+)$ is commutative and associative (commutative semigp).
- $(K,\cdot)$ is an abelian group ← a field is not a semifield.
- distributive law.

Eg. $\mathbb{R}_{>0}$

$$\mathbb{R}(t)_{>0} := \left\{ t^e \frac{f_1(t)}{f_2(t)} \;\middle|\; \begin{array}{l} e\in\mathbb{Z},\ f_1(t), f_2(t) \in \mathbb{R}[t] \\ \text{s.t. } f_1(0), f_2(0) \in \mathbb{R}_{>0} \end{array} \right\}$$

Tropical $(\mathbb{Z}, \oplus, \odot)$.

$a \oplus b = \min\{a,b\}$

$a \odot b = a+b$

$\mathbb{R}(t)_{>0} \twoheadrightarrow \mathrm{Trop}\,\mathbb{Z}$ semifield homomorphism

$f \mapsto$ deg of lowest term

Remark. $\mathbb{R}(t)_{>0}' = \{$ coeff of highest term $>0 \}$

$\mathrm{Trop}\,\mathbb{Z}'$: $a \oplus' b = \max\{a,b\}$

Then $\mathbb{R}(t)_{>0}' \twoheadrightarrow \mathrm{Trop}\,\mathbb{Z}'$

Remark. $\mathbb{R}\{\{t\}\} = \bigcup_{n\in\mathbb{N}} \mathbb{R}(t^{1/n})$.

Puiseux series

Define $\mathbb{R}\{\{t\}\}_{>0} = \bigcup_{n\in\mathbb{N}} \mathbb{R}(t^{1/n})_{>0}$

Then

$\mathbb{R}\{\{t\}\}_{>0} \twoheadrightarrow \mathrm{Trop}\,\mathbb{Q}$

eg. $\mathbb{k}_1 = \{1\}$. $1+1=1$. $1\cdot 1 = 1$.

Any semifield k have $k \to k_1$ semifield homo

Upshot: $G(k) \xrightarrow[\text{base change}]{} G(k')$ ← skeleton of $G(k)$