MMAT 5011 Analysis II
2016-17 Term 2
Assignment 6 (Optional)

This assignment is optional. You do not have to turn in it. However, you are encouraged to try all the problems.

1. Consider $P_1(\mathbb{R})$ as a subspace of $L^2[0,1]$. Let $f : P_1(\mathbb{R}) \to \mathbb{R}$ be defined by $f(p) = p'(0)$. Find a polynomial $q \in P_1(\mathbb{R})$ such that $f(p) = \langle p, q\rangle$ for all $p \in P_1(\mathbb{R})$.

2. Prove that $(\alpha T)^* = \overline{\alpha} T^*$ for a bounded linear operator $T : H_1 \to H_2$ between Hilbert spaces and a scalar α.

3. Let $T : H_1 \to H_2$ be a bounded operator. Show that $N(T) = [T^*(H_2)]^\perp$. Here $N(T)$ and $T^*(H_2)$ is the null space of T and the range of T^* respectively.

4. Suppose that $T : H \to H$ is a bounded linear opeartor. Let
$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_2 = \frac{1}{2i}(T - T^*)$$
   (a) Show that T_1 and T_2 are self adjoint.
   (b) Show that $T = T_1 + iT_2$.
   (c) Show that if $S_1, S_2 : H \to H$ are self-adjoint operators such that $T = S_1 + iS_2$, then $S_1 = T_1$ and $S_2 = T_2$.

5. Let H be a finite dimensional complex inner product space. Suppose a linear operator $T : H \to H$ satisfies $T^* = -T$. Show that T has an orthonormal eigenbasis (i.e. an orthonormal basis consisting of eigenvectors) and all its eigenvalues are purely imaginary.

6. Suppose $T_n, T : H_1 \to H_2$ are bounded. Show that if $T_n \to T$, then $T_n^* \to T^*$.

7. Show that $I + T^*T : H \to H$ is injective for a bounded linear operator $T : H \to H$.

8. Suppose that $T : H \to H$ is a normal operator and α is a scalar.
   (a) Show that $\|T(x)\| = \|T^*(x)\|$.
   (b) Show that $T - \alpha I : H \to H$ is a normal operator. Here $I : H \to H$ is the identity operator $I(x) = x$.
   (c) Show that $T(x) = \alpha x$ if and only if $T^*(x) = \overline{\alpha} x$.
   (d) Show that if x, y are eigenvectors of T corresponding to different eigenvalues, then x and y are orthogonal.