MMAT 5011 Analysis II
2016-17 Term 2
Assignment 2
Due date: Feb 21, 2017

Assume $\mathbb{F} = \mathbb{R}$ in all the following problems.

You do not have to turn in the solution of optional problems. However, you are encouraged to try all the problems.

1. (a) Let A be a measure zero subset of $\mathbb{R}$ and $B \subset A$. Show that B is measure zero.

   (b) Let A_i be a measure zero subset of $\mathbb{R}$ for each $i \in \mathbb{N}$. Show that the countable union $\bigcup_{i=1}^{\infty} A_i$ is measure zero.

2. Suppose $\{x_1, x_2, \ldots, x_n\}$ is a basis of a real normed space X. Show that

$$\{c_1 x_1 + c_2 x_2 + \ldots + c_n x_n : c_i > 0 \text{ for } 1 \le i \le n\}$$

   is an open subset of X. (Hint: Use lemma 2.4-1.)

3. A series $\sum_{i=1}^{\infty} x_i$ in a normed space is said to be

   - convergent if $s_k = \sum_{i=1}^{k} x_i$ is a convergent sequence.
   - absolute convergent if $\sigma_k = \sum_{i=1}^{k} \|x_i\|$ is a convergent sequence.

   (a) Show that an absolute convergent series in a Banach space X is convergent.

   (b) (Optional) Give a counter-example to show that the implication in (a) is no longer true if X is an incomplete normed space.

4. Consider the following subspaces of the normed space l^{∞}:

$$Y = \{(x_1, x_2, \ldots) : \exists N > 0 \text{ such that } x_i = 0 \ \forall i \ge N\}$$
$$c_0 = \left\{(x_1, x_2, \ldots) : \lim_{i \to \infty} x_i = 0\right\}$$

   (a) Is Y complete? (Hint: Do $x_n \in Y$ and $x_n \to x \in l^{\infty}$ imply that $x \in Y$?)

   (b) (Optional) Is c_0 complete?

5. Let $T : X \to Y$ be a linear operator between normed spaces X and Y. Show that T is bounded if and only if $T(A) = \{T(x) : x \in A\}$ is bounded for any bounded subset $A \subset X$.

6. Give an example of a linear operator $T : X \to X$ on a normed space X with $\|T\| = 1$ and $\|T(x)\| < \|x\|$ for all non-zero $x \in X$.

7. Let X be the normed space of polynomials with norm given by

$$\|p\| = \int_0^1 |p(x)|dx$$

Show that differential operator $T : X \to X$ defined by $T(p) = p'$ is unbounded.

8. (Optional) Show that the interval $[0, 1]$ is not measure zero.

9. (Optional) Consider the vector space $C[-1, 1]$ of real continuous functions on the interval $[-1, 1]$. Let $f_n \in C[-1, 1]$ be defined by

$$f_n(n) = \begin{cases} 0 & \text{if } -1 \leq x < 0; \\ nx & \text{if } 0 \leq x \leq \frac{1}{n}; \\ 1 & \text{if } \frac{1}{n} < x \leq 1. \end{cases}$$

Determine whether the sequence (f_n) is Cauchy and/or convergent under the following norms on $C[-1, 1]$.

(a) $\|f\|_\infty = \sup_{-1 \leq x \leq 1} |f(x)|$;

(b) $\|f\|_1 = \int_{-1}^1 |f(x)|dx$.