MATH2010 Advanced calculus, 2020-21
HOMEWORK ONE
Suggested Solution

1. By putting coordinates, the required angle θ is the angle between the vector $u = (0,0,1)$ and the vector $v = (1,1,1)$.

   So $\cos\theta = \dfrac{u \cdot v}{|u||v|} = \dfrac{1}{\sqrt{3}} < \dfrac{1}{\sqrt{2}} = \cos\dfrac{\pi}{4}$.

   Since cos is decreasing on the interval $\left[0, \dfrac{\pi}{2}\right]$, we conclude that $\theta > \dfrac{\pi}{4}$.

   Remark: A few students conclude that $\theta < \dfrac{\pi}{4}$ in the last step, missing the fact that cos is decreasing on that interval.

2. (a) Suppose $P_2 = (x_2, y_2, z_2)$ be a point on the plane. Hence we have $Ax_2 + By_2 + C_2 = D$.
   The unit normal of the plane is $N = \frac{(A,B,C)}{\sqrt{(A^2+B^2+C^2)}}$.
   Hence the distance d from P_1 to the plane is

$$
\begin{aligned}
d &= | < P_2P_1, N > | \\
&=|\frac{A(x_2 - x_1) + B(x_2 - x_1) + C(x_2 - x_1)}{\sqrt{(A^2 + B^2 + C^2)}}| \\
&=\frac{|(Ax_2 + Bx_2 + Cx_2) - (Ax_1 + Bx_1 + Cx_1)|}{\sqrt{(A^2 + B^2 + C^2)}} \\
&=\frac{|Ax_1 + Bx_1 + Cx_1 - D|}{\sqrt{(A^2 + B^2 + C^2)}}.
\end{aligned} \tag{1}
$$

   (b) By solving the equation $2x - y = 3x - z = 0$, we may assume the center $P = (3t, 2t, t)$. Since the sphere is tangent to the planes $x + y + z = 3$ and $x + y + z = 9$, the distance from P to the two planes must equal, i.e.

$$
\frac{|3t + 2t + t - 3|}{\sqrt{(1 + 1 + 1)}} = \frac{|3t + 2t + t - 9|}{\sqrt{(1 + 1 + 1)}}. \tag{2}
$$

   Hence we obtain $t = 1$. Since the sphere is tangent the plane $x + y + z = 3$, the distance is just the radius, $r = \frac{|3+2+1-3|}{\sqrt{(1+1+1)}} = \sqrt{3}$.
   Therefore, we have the equation $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 3$.

3. (a) The velocity $v(t) = r'(t) = (-5\sin t)\hat{\mathbf{j}} + (3\cos t)\hat{\mathbf{k}}$.
   The acceleration $a(t) = v'(t) = (-5\cos t)\hat{\mathbf{j}} + (-3\sin t)\hat{\mathbf{k}}$.
   Then $0 =< v(t), a(t) >$ is equivalent to $16\sin t\cos t = 8\sin 2t = 0$.
   The solution is $t = 0, \frac{\pi}{2}$, or π.

   (b) $\displaystyle\int_0^\pi |v(t)|\,dt = \int_0^\pi \sqrt{(-5\sin t)^2 + (3\cos t)^2}\,dt$.

4. (a)

$$\begin{aligned}
&\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y - xy + 4x^2 - 4x} \\
=&\lim_{(x,y)\to(2,-4)} \frac{y+4}{(x^2-x)(y+4)} \\
=&\lim_{(x,y)\to(2,-4)} \frac{1}{x^2-x} \\
=&\frac{1}{2}
\end{aligned} \tag{3}$$

(b) $\lim_{(x,y)\to(0,0)} |2x^2+y^2| = 0.$

Hence, by squeeze theorem, $\lim_{(x,y)\to(0,0)} (2x^2+y^2)\sin\frac{1}{\sqrt{x^2+4y^2}} = 0$

(c) Along the path $y^2 = kx^5$,

$$\begin{aligned}
&\lim_{(x,y)\to(0,0),y^2=kx^5} \frac{x^5y^2}{x^{10}-y^4} \\
=&\lim_{x\to 0} \frac{kx^{10}}{x^{10}-k^2x^{10}} \\
=&\frac{k}{1-k^2}
\end{aligned} \tag{4}$$

This limit depends on the path taken, so the limit does not exist.

(d) Along $x = 1$,

$$\begin{aligned}
&\lim_{(x,y)\to(1,-1),x=1} \frac{xy+1}{x^2-y^2} \\
=&\lim_{y\to -1} \frac{y+1}{1-y^2} \\
=&\lim_{y\to -1} \frac{1}{1-y} \\
=&\frac{1}{2}
\end{aligned} \tag{5}$$

Along $y = -1$,

$$\begin{aligned}
&\lim_{(x,y)\to(1,-1),y=-1} \frac{xy+1}{x^2-y^2} \\
=&\lim_{x\to 1} \frac{-x+1}{x^2-1} \\
=&\lim_{x\to 1} \frac{-1}{x+1} \\
=&-\frac{1}{2}
\end{aligned} \tag{6}$$

The two limits do not agree, so the limit does not exist.

(e) Along $x = 0$,

$$\begin{aligned} &\lim_{(x,y)\to(0,0),x=0} \frac{2x}{x^2+x+y^2} \\ =&0 \end{aligned} \tag{7}$$

Along $y = 0$,

$$\begin{aligned} &\lim_{(x,y)\to(0,0),y=0} \frac{2x}{x^2+x+y^2} \\ =&\lim_{x\to 0} \frac{2x}{x^2+x} \\ =&\lim_{x\to 0} \frac{2}{x+1} \\ =&2 \end{aligned} \tag{8}$$

The two limits do not agree, so the limit does not exist.