MATH2010 Advanced Calculus I, 2020-21

HOMEWORK TWO

Due 3pm Monday, Oct. 26

Q1. If $w(x,t)$ represents the temperature at position x at time t in a uniform wire with perfectly insulated sides, then the partial derivatives w_{xx} and w_t satisfy a differential equation of the form

$$w_{xx} = \frac{1}{c^2} w_t.$$

This equation is called the *one-dimensional heat equation.* The value of the positive constant c^2 is determined by the material from which the wire is made.

(a) Find all solutions of the one-dimensinoal heat equation of the form $w = e^{rt}\sin \pi x$, where r is a constant.

(b) Find all solutions of the one-dimensinoal heat equation of the form $w = e^{rt}\sin kx$, and satisfy the conditions that $w(0,t) = 0$ and $w(L,t) = 0$. What happens to these solutions as $t \to \infty$?

Q2. Determine whether the following functions are continuous, differentiable, or not.

(a) $f(x,y) = x\sin y$.

(b) $f(x,y) = |xy|$.

(c) $f(x,y) = \begin{cases} xy\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(d) $f(x,y) = \begin{cases} \frac{x^2y^2}{(x^2+y^2)^{3/2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Q3. (Optional, no need to hand in) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Fix a point (x_0, y_0). Consider a linear approximation of $f(x,y)$ near (x_0,y_0), defined by

$$l(x,y) = f(x_0,y_0) + r(x - x_0) + s(y - y_0),$$

where $r, s \in \mathbb{R}$ are constants. Suppose the error

$$\epsilon(x,y) = f(x,y) - f(x_0,y_0) - r(x - x_0) - s(y - y_0)$$

satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\epsilon(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Show that the partial derivatives $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist with $r = f_x(x_0,y_0)$ and $s = f_y(x_0,y_0)$.

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