MATH2040C Linear Algebra II
2017-18 Solution to Homework 2

Exercise 2.A

5 (a) For $\alpha, \beta \in \mathbb{R}$, consider the equation

$$\alpha(1+i) + \beta(1-i) = 0.$$

The real part and imaginary part of L.H.S. are $\alpha + \beta$ and $\alpha - \beta$ respectively and they must be zero. The only solution is $\alpha = \beta = 0$. This shows that $(1+i, 1-i)$ is linearly independent over $\mathbb{R}$.

(b) Note that $i \cdot (1+i) + 1 \cdot (1-i) = 0$. Since $i, 1$ are not zero, $(1+i, 1-i)$ is linearly dependent over $\mathbb{C}$.

9* Consider the real vector space $\mathbb{R}$. Take $m = 1$, $v_1 = 1$, $w_1 = -1$. Since (v_1) and (w_1) are lists with only one element which is also non-zero, they are linearly independent lists of vectors in V. However $v_1 + w_1 = 0$ and $\{0\}$ is not linearly independent since $1 \cdot 0 = 0$ but $1 \neq 0$. Therefore the statement is false.

10 Given that $(v_1 + w, \ldots, v_m + w)$ is linearly dependent, there exists $a_1, \ldots, a_m \in \mathbb{F}$ not all zero such that $a_1(v_1 + w) + \cdots + a_m(v_m + w) = \overrightarrow{0}$. After rearranging the sum, we have $a_1 v_1 + \cdots + a_m v_m = -(a_1 + \cdots + a_m)w$. Let $c = -(a_1 + \cdots + a_m)$. If $c = 0$, then $a_1 v_1 + \cdots + a_m v_m = \overrightarrow{0}$ and this implies $(v_1, \ldots, v_m)$ being linearly dependent. Contradiction. Therefore $c \neq 0$ and $\frac{a_1}{c} v_1 + \cdots + \frac{a_m}{c} v_m = w$. This implies $w \in \operatorname{span}(v_1, \ldots, v_m)$.

11 ($\Rightarrow$) Assume $(v_1, \ldots, v_m, w)$ is linearly independent but $w \in \operatorname{span}(v_1, \ldots, v_m)$. Then there exists $a_1, \ldots, a_m \in \mathbb{F}$ not all zero such that $a_1 v_1 + \cdots + a_m v_m = w$. Rewriting the equation, we have $a_1 v_1 + \cdots + a_m v_m + (-1)w = \overrightarrow{0}$. Since $-1 \neq 0$, $(v_1, \ldots, v_m, w)$ cannot be linearly independent. Contradiction. Therefore $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

($\Leftarrow$) Assume $w \notin \operatorname{span}(v_1, \ldots, v_m)$ but $\{v_1, \ldots, v_m, w\}$ is linearly dependent. Then there exists $a_1, \ldots, a_m, c \in \mathbb{F}$ not all zero such that $a_1 v_1 + \cdots + a_m v_m + cw = \overrightarrow{0}$. If $c = 0$, then $a_1, \ldots, a_m$ cannot be all zero. However, this implies $(v_1, \ldots, v_m)$ is linearly dependent. Contradiction. If $c \neq 0$, then we have $\frac{-a_1}{c} v_1 + \cdots + \frac{-a_m}{c} v_m = w$. Therefore $w \in \operatorname{span}(v_1, \ldots, v_m)$. Contradiction. Hence $(v_1, \ldots, v_m, w)$ is linearly independent.

12 Given any six polynomials $(f_j(x) = a_{0j} + a_{1j}x + \cdots + a_{4j}x^4)_{j=1}^6$ of degree at most 4, consider the following equation

$$y_1 f_1(x) + \cdots y_6 f_6(x) = \overrightarrow{0} \tag{1}$$

where $y_1, \ldots, y_6 \in \mathbb{F}$. By comparing coefficients, we have

$$\sum_{j=1}^{6} y_j a_{ji} = 0 \quad \forall j = 0, \ldots, 4.$$

This is a homogeneous system of 5 linear equations in 6 unknowns. Therefore it has a nontrivial solution, say $(y_1, \ldots, y_6) = (c_1, \ldots, c_6)$ where c_i are not all zero. This implies the corresponding equation (2) is a non-trivial linear combination of $\overrightarrow{0}$ and the polynomials cannot be linearly independent.

15* Assume not, there exists a finite subset $S = \{v_1, \dots, v_n\} \subset \mathbb{F}^\infty$ such that $\operatorname{span} S = V$ (S cannot be empty since $\mathbb{F}^\infty$ contains a non-zero vector. For any $i \in \mathbb{N}$, let e_i be the sequence such that the only non-zero entry is the ith-entry with value 1. Since $e_i \in \mathbb{F}^\infty$, there exists $c_{ij} \in \mathbb{F}$ for all i, j such that

$$\sum_{j=1}^{n} c_{ij} v_j = e_i \quad \forall i \in \mathbb{N}.$$

Now consider the following homogeneous system of linear equations:

$$\begin{cases} c_{11}x_1 + c_{21}x_2 + \cdots + c_{(n+1)1}x_{n+1} & = 0 \\ \vdots & \\ c_{1n}x_1 + c_{2n}x_2 + \cdots + c_{(n+1)n}x_{n+1} & = 0 \end{cases}$$

It has n equations with $n+1$ unknowns. So there exists a nontrivial solution $(x_1, \dots, x_{n+1})$. However,

$$\sum_{i=1}^{n+1} x_i e_i = \sum_{i=1}^{n+1} x_i \left(\sum_{j=1}^{n} c_{ij} v_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n+1} x_i c_{ij} \right) v_j = \sum_{j=1}^{n} 0 v_j = \overrightarrow{0}$$

By comparing entries, $x_i = 0$ for all i, which is impossible by construction. Contradiction. Therefore $\mathbb{F}^\infty$ is infinite dimensional.

17 Since $p_j(2) = 0$ for each j, by division algorithm, $p_j(x) = (x-2)q_j(x)$ for some polynomial q_j in $\mathcal{P}_{m-1}(F)$ for each j. Using similar argument in Problem 12, $(q_0, \dots, q_m)$ is linearly dependent in $\mathcal{P}_{m-1}(F)$. Therefore there exists $a_0, \dots, a_m \in \mathbb{F}$ not all zero such that $a_0 q_0 + \cdots a_m q_m = \overrightarrow{0}$. By multiplying $(x-2)$ on both sides, we have $a_0 p_0 + \cdots a_m p_m = \overrightarrow{0}$, which is a nontrivial linear combination of $\overrightarrow{0}$.

Exercise 2.B

3* (a) Let $v_1 = (3,1,0,0,0), v_2 = (0,0,7,1,0), v_3 = (0,0,0,0,1)$. For any $x = (x_1, x_2, x_3, x_4, x_5) \in U$, $x = x_2 v_1 + x_4 v_2 + x_5 v_3$. So $\beta = (v_1, v_2, v_3)$ spans U. If $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 v_1 + a_2 v_2 + a_3 v_3 = \overrightarrow{0}$, by comparing the 2nd, 4th, 5th entries, a_1, a_2, a_3 are all zero. Therefore β is linearly independent and is a basis of U.

(b) Let $v_4 = (1,0,0,0,0), v_5 = (0,0,1,0,0)$ and consider $\beta' = (v_1, v_2, v_3, v_4, v_5)$. If a_1, a_2, a_3, a_4, $a_5 \in \mathbb{R}$ such that $a_1 v_1 + \ldots + a_5 v_5 = \overrightarrow{0}$, by above method, a_1, a_2, a_3 are all zero. So $a_4 v_4 + a_5 v_5 = \overrightarrow{0}$. By comparing the 1st and 3rd entries, a_4, a_5 are all zero. Therefore β' is linearly independent. For any $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, note that $x = x_2 v_1 + x_4 v_2 + x_5 v_3 + (x_1 - 3x_2) v_4 + (x_3 - 7x_4) v_5$ and the coefficients of v_i are in $\mathbb{R}$. Therefore β' spans $\mathbb{R}^5$ and is a basis of $\mathbb{R}^5$ extending β.

(c) Let $W = \operatorname{span}(v_4, v_5)$. By argument in the proof of Theorem 2.34 in the textbook, $\mathbb{R}^5 = U \oplus W$.

5 Consider $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 + x^3$, $p_3(x) = x^3$. They are in $\mathcal{P}_3(\mathbb{F})$ and none of them has degree 2. For any $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in \mathcal{P}_3(\mathbb{F})$ with $a_i \in \mathbb{F}$, suppose there exists $b_0, \dots, b_3 \in \mathbb{F}$ such that $f(x) = b_0 p_0(x) + \cdots + b_3 p_3(x)$. By comparing the coefficients of 1, x, x^2, we must have $b_0 = a_0, b_1 = a_1, b_2 = a_2$. Now $f(x) - b_2p_2(x) - b_1p_1(x) - b_0p_0(x) = a_3x^3 - b_2x^3$. Take $b_3 = a_3 - b_2 = a_3 - a_2$ and we can see that every polynomial in $\mathcal{P}_3(\mathbb{F})$ can be written uniquely in the form $f(x) = b_0p_0(x) + \cdots + b_3p_3(x)$. Thus it is a basis by Criterion 2.29 in the textbook.

6 Let $v \in V$. Since v_1, v_2, v_3, v_4 is a basis of V, there exist unique $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$. Therefore $v = a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2 + a_1)(v_3 + v_4) + (a_4 - a_3 + a_2 - a_1)v_4$. Assume there exist $b_1, b_2, b_3, b_4 \in \mathbb{F}$ such that $v = b_1(v_1+v_2)+b_2(v_2+v_3)+b_3(v_3+v_4)+b_4v_4$. Then $v = b_1v_1+(b_1+b_2)v_2+(b_2+b_3)v_3+(b_3+b_4)v_4$. By uniqueness, $b_1 = a_1$, $b_1 + b_2 = a_2$, $b_2 + b_3 = a_3$, $b_3 + b_4 = a_4$. By repeated substitution, we see that $b_2 = a_2 - a_1$, $b_3 = a_3 - a_2 + a_1$, $b_4 = a_4 - a_3 + a_2 - a_1$. Therefore the expression of v as a linear combination of $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is unique and $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is a basis by Criterion 2.29 in the textbook.

7* Consider $V = \mathbb{R}^4$, $v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 0), v_4 = (0, 0, 0, 1), U = \{(x_1, x_2, x_3, x_4) \in V : x_3 = x_4\}$. From Example 2.28 of textbook, v_1, v_2, v_3, v_4 is a basis of V. Since $0 \neq 1$, $v_1, v_2 \in U$ and $v_3, v_4 \notin U$. U is a subspace of V. However, v_1, v_2 is not a basis of U since $(0, 0, 1, 1) \in U$ but it is not in the span of v_1, v_2, which is $\{(x_1, x_2, 0, 0) : x_1, x_2 \in \mathbb{R}\}$.

8 Let $v \in V$, then there exist $u \in U$, $w \in W$ such that $v = u + w$. Since $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W, there exists $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ such that

$$\sum_{i=1}^{m} a_iu_i = u \quad \text{and} \quad \sum_{j=1}^{n} b_jw_j = w.$$

Therefore

$$v = u + w = \sum_{i=1}^{m} a_iu_i + \sum_{j=1}^{n} b_jw_j$$

and $u_1, \ldots, u_m, w_1, \ldots, w_n$ span V.

Now suppose there exists $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ such that

$$\sum_{i=1}^{m} a_iu_i + \sum_{j=1}^{n} b_jw_j = \overrightarrow{0}.$$

By rearrange the terms, we have

$$\sum_{i=1}^{m} a_iu_i = \sum_{j=1}^{n} -b_jw_j.$$

Now L.H.S. is in U and R.H.S. is in W. Therefore both are in $U \cap W = \{\overrightarrow{0}\}$. So

$$\sum_{i=1}^{m} a_iu_i = \overrightarrow{0} \text{ and } \sum_{j=1}^{n} b_jw_j = \overrightarrow{0}.$$

Since $u_1, \ldots, u_m$ is linearly independent and $w_1, \ldots, w_n$ is also linearly independent, we have $a_1, \ldots, a_m, b_1, \ldots, b_n$ all zero. Thus $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent.

Exercise 2.C

1 Let $(u_1, \ldots, u_m)$ be a basis of U, where $m = \dim U$. It is a linearly independent subset of V. By Theorem 2.39 in the textbook, it is a basis of V since its length is equal to $\dim U = \dim V$. Therefore $V = \operatorname{span}(u_1, \ldots, u_m) = U$.

5* (a) Consider $v_0(x) = 1$, $v_1(x) = x - 6$, $v_3(x) = (x-6)^3$, $v_4(x) = (x-6)^4$. Let $y = x - 6$, by change of variable, they are just powers of y and thus linearly independent. Suppose $p \in \mathcal{P}_4(\mathbb{R})$ such that $p''(6) = 0$. Write $q(y) = p(y+6) = p(x) = a_0 + a_1 y + \cdots + a_4 y^4$, then $q''(b) = p''(b+6)$. Therefore $2a_2 = q''(0) = p''(6) = 0$. This implies $q(y)$ is a linear combination of $1, y, y^3, y^4$. Changing back the variables, we have $p(x)$ being a linear combination of v_0, v_1, v_3, v_4, which is therefore a basis of U.

(b) From the above discussion, adding $v_2(x) = (x-6)^2$ to the list (v_0, v_1, v_3, v_4) is a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Let $W = \operatorname{span}\{v_2\}$. Then by the argument in the proof of Theorem 2.34 in the textbook, $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

7 (a) We claim that $U = \operatorname{span}(1, (x-2)(x-5)(x-6), x(x-2)(x-5)(x-6))$. For $p \in U$, let $s := p - p(2) \in \mathcal{P}_4(\mathbb{F})$. It satisfies $s(2) = s(5) = s(6) = 0$. By division algorithm, $s(x) = q(x)(x-2)(x-5)(x-6)$ for some polynomial of degree at most 1. Therefore $v_1(x) = 1$, $v_2(x) = (x-2)(x-5)(x-6)$, $v_3(x) = x(x-2)(x-5)(x-6)$ spans U. Since they have different degree, by comparing the coefficient of the highest degree terms in $a_1 v_1 + a_2 v_2 + a_3 v_3 = \overrightarrow{0}$ successively, where $a_1, a_2, a_3 \in \mathbb{F}$, we have $a_1 = a_2 = a_3 = 0$. By direct computation, they contained in U. Therefore v_1, v_2, v_3 is a basis of U.

(b) Let $v_4(x) = x$, $v_5(x) = x^2$. Consider the sum

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 = \overrightarrow{0}$$

where $a_1, \ldots, a_5 \in \mathbb{F}$. By comparing the coefficient again, $(v_1, \ldots, v_5)$ is a linearly independent list with length equal to the dimension of the vector space. Hence by Theorem 2.39 in the textbook it is a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Take $W = \operatorname{span}\{v_4, v_5\}$. Then by the argument in the proof of Theorem 2.34 in the textbook, $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

9 If $(v_1 + w, \ldots, v_m + w)$ is linearly independent, it is a basis for $U := \operatorname{span}(v_1 + w, \ldots, v_m + w)$. Therefore its dimension is $m > m - 1$.

If $(v_1 + w, \ldots, v_m + w)$ is linearly dependent, by Exercise 2.A Problem 10, $w \in X := \operatorname{span}(v_1, \ldots, v_m)$. So there exist unique $b_1, \ldots, b_m \in \mathbb{F}$ such that

$$w = b_1 v_1 + \cdots + b_m v_m. \tag{2}$$

If all $b_i = 0$, then $w = 0$ and $(v_1 + w, \ldots, v_m + w) = (v_1, \ldots, v_m)$ which is linearly independent. Contradiction. So there exists $b_i \neq 0$. WLOG, assume $b_1 \neq 0$. Suppose $a_2, \ldots, a_m \in \mathbb{F}$ such that

$$\sum_{i=2}^{m} a_i (v_i + w) = \overrightarrow{0}\,.$$

By substituting w using Equation (2), we have

$$\lambda b_1 v_1 + \sum_{i=2}^{m}(a_i + \lambda b_i)v_i = \overrightarrow{0}$$

where $\lambda = \sum_{i=2}^{m} a_i$. By linear independence, $a_i = -\lambda b_i$ for $i = 2, \ldots, m$ and $\lambda b_1 = 0$. Since $b_1 \neq 0$, $\lambda = 0$ and all a_i are zero. Therefore $(v_2 + w, \ldots, v_m + w)$ is linearly independent and

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq \dim \operatorname{span}(v_2 + w, \ldots, v_m + w) = m - 1.$$

10* Since $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$, by Theorem 2.39 in the textbook, it suffices to check $(p_0, \ldots, p_m)$ is linearly independent, which will be done by induction.

Consider p_0, since p_0 has degree 0, it is non-zero since degree of the zero polynomial is $-\infty$. Therefore p_0 is linearly independent.

Suppose $(p_0, \ldots, p_k)$ is linearly independent for some non-negative integer k.
Consider $(p_0, \ldots, p_{k+1})$. Suppose $a_0, \ldots, a_{k+1} \in \mathbb{F}$ such that $a_0 p_0 + \cdots + a_{k+1} p_{k+1} = \overrightarrow{0}$. Consider the coefficient of the term x^{k+1}. Since $p_0, \ldots, p_k$ has degree less than $k + 1$, their coefficient of x^{k+1} are 0. Now the coefficient of the term x^{k+1} on the L.H.S. is a_{k+1} times the coefficient of x^{k+1} in p_{k+1}, which is non-zero since degree of p_{k+1} is $k+1$. Therefore $a_{k+1} = 0$ and by induction hypothesis $(p_0, \ldots, p_k)$ is linearly independent and $a_0 = \cdots = a_k = 0$. Hence $(p_0, \ldots, p_{k+1})$ is linearly independent.

12 By Theorem 2.43 in the textbook, $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$. Since $U + W \subset \mathbb{R}^9$, $\dim(U + W) \leq \dim \mathbb{R}^9 = 9$. Therefore

$$\dim(U \cap W) = \dim U + \dim W - \dim(U + W) \geq 5 + 5 - 9 = 1.$$

Since $\dim\{\overrightarrow{0}\} = 0 \neq 1$, $U \cap W \neq \{\overrightarrow{0}\}$.

14* The case $m = 1$ is true because U_1 is given to be finite dimensional and L.H.S. is equal to R.H.S. Assume the inequality is true for some positive integer k. Let $U_1, \ldots, U_k, U_{k+1}$ be finite-dimensional subspaces of V. By assumption $W := U_1 + \cdots + U_k$ is finite dimensional. By the proof of Theorem 2.43 in the textbook, $U_1 + \cdots + U_k + U_{k+1} = W + U_{k+1}$ is again finite-dimensional and $\dim(W + U_{k+1}) \leq \dim W + \dim U_{k+1}$. By assumption again $\dim W \leq \dim U_1 + \cdots + \dim U_k$. Therefore

$$\dim(U_1 + \cdots + U_k + U_{k+1}) \leq \dim(U_1 + \cdots + U_k) + \dim U_{k+1} \leq \dim U_1 + \cdots + \dim U_k + \dim U_{k+1}.$$

17 Consider $V = \mathbb{R}^3$, $U_1 = \{(x, y, y) : x, y \in \mathbb{R}\}$, $U_2 = \{(y, x, y) : x, y \in \mathbb{R}\}$, $U_3 = \{(y, y, x) : x, y \in \mathbb{R}\}$. They are subspace of V of dimension 2 (using Example 2.28(e) in the textbook). Note that

$$U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U_1 \cap U_2 \cap U_3 = \{(x, x, x) : x \in \mathbb{R}\} =: W.$$

It is because for any $v \in U_1 \cap U_2$, $v = (x_1, y_1, y_1) = (y_2, x_2, y_2)$ for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Therefore $x_1 = y_2$, $y_1 = x_2$, $y_1 = y_2$ and all entries are equal. So $U_1 \cap U_2 = \{(x, x, x) : x \in \mathbb{R}\}$. The reverse inclusion is clear and the other equalities are proven similarly. Also, $U_1 + U_2 + U_3 =$

V since for any $w = (a, b, c) \in V$, $w = (a, 0, 0) + (0, b, 0) + (0, 0, c)$ where these three terms contain in U_1, U_2, U_3 respectively. Hence $\dim(U_1 + U_2 + U_3) = 3$. However, $\dim W = 1$ since $(1, 1, 1)$ is a basis of W. Therefore R.H.S. of the equation is $2+2+2-1-1-1+1 = 4 \neq 3$.

(The problem of this analogy is that $U_1 \cap (U_2 \cup U_3) = (U_1 \cup U_2) \cap (U_1 \cup U_3)$ for sets but $U_1 \cap (U_2 + U_3)$ is not equal to $(U_1 + U_2) \cap (U_1 + U_3)$ for vector subspaces in general.)