**MATH 2040C Linear Algebra II**

**2017-18 Term 2**

**Midterm 2**

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NAME:________________________ ID:________________________

**Instruction**: Answer ALL questions and show your work with explanation.

**Time**: 60 minutes

| Question | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| Total | /40 |

1. (**True or False**) Please circle the correct answer. Each question is worth 1 point.

(a) A list of orthonormal vectors $v_1, v_2, \ldots v_n$ in an inner product space is linearly independent.

**TRUE** **FALSE**

(b) Let $T : V \to W$ be an isomorphism between finite dimensional real vector spaces $V$ and $W$. Then for any ordered bases $\alpha$ of $V$ and $\beta$ of $W$, $\mathcal{M}(T, \alpha, \beta)$ is a square matrix and is invertible.

**TRUE** **FALSE**

(c) Let $V$ be a finite dimensional vector space. For any diagonalizable $S, T \in \mathcal{L}(V)$, $S + T \in \mathcal{L}(V)$ is also diagonalizable.

**TRUE** **FALSE**

(d) Let $T$ be a linear operator on a vector space $V$. Then the set of eigenvectors corresponding to an eigenvalue of $T$ is a subspace of $V$.

**TRUE** **FALSE**

(e) For any linear operator $T$ on $\mathbb{R}^7$, there exists an ordered basis $\beta$ of $\mathbb{R}^7$ such that $\mathcal{M}(T, \beta)$ is upper triangular.

True if $\mathbb{F} = \mathbb{C}$ (Schur's Theorem) **TRUE** **FALSE**

(f) For any finite dimensional complex vector spaces $V$ and $W$, the complex vector spaces $\mathcal{L}(V, W)$ and $\mathcal{L}(W, V)$ are isomorphic.

Same dimension $\Rightarrow$ isomorphic **TRUE** **FALSE**

(g) Every finite dimensional inner product space has an orthonormal basis.

**TRUE** **FALSE**

(h) Let $V$ be a real inner product space and $v, w \in V$. Then $\|v + w\| = \|v\| + \|w\|$ if and only if there exists a real number $c$ such that $v = cw$ or $w = cv$.

$c > 0$ **TRUE** **FALSE**

2. (8 pts) Answer the following questions.

(a) Let $V$ be a real inner product space and $v, w \in V$. Show that

$$\langle v, w \rangle = \frac{1}{4}(\|v+w\|^2 - \|v-w\|^2).$$

$$\text{R.H.S.} = \frac{1}{4}\left( \langle v+w, v+w \rangle - \langle v-w, v-w \rangle \right)$$

$$= \frac{1}{4}\begin{pmatrix} \langle v,v \rangle + \langle v,w \rangle + \langle w,v \rangle + \langle w,w \rangle \\ - \langle v,v \rangle + \langle v,w \rangle + \langle w,v \rangle - \langle w,w \rangle \end{pmatrix}$$

$$= \frac{1}{4}\left( 2\langle v,w \rangle + 2\overline{\langle v,w \rangle} \right)$$

$$= \langle v,w \rangle \quad , \text{ since } \langle v,w \rangle \in \mathbb{F} = \mathbb{R}$$

(b) Let $V$ be a finite dimensional real vector space and $T \in \mathcal{L}(V)$. Suppose $v, w \in V$ are non-zero vectors such that $T(v) = 4w$ and $T(w) = 4v$. Show that $T$ has at least one eigenvalue.

Soln 1: $T(v+w) = T(v) + T(w) = 4w + 4v = 4(v+w)$

$T(v-w) = T(v) - T(w) = 4w - 4v = -4(v-w)$

$v, w \neq \vec{0} \Rightarrow v+w$ or $v-w \neq \vec{0}$

$\Rightarrow 4$ or $-4$ is an eigenvalue of $T$

Soln 2: $T^2(v) = T(T(v)) = T(4w) = 4T(w) = 4(4v) = 16v$

$\Rightarrow (T^2 - 16I)(v) = 0$

$\Rightarrow (T-4I)(T+4I)$ is not injective

$\Rightarrow T-4I$ or $T+4I$ is not injective

$\Rightarrow 4$ or $-4$ is an eigenvalue of $T$

3. (9 pts) Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all real polynomials of degree at most 2 and $\beta = \{1, x, x^2\}$ be an ordered basis of $\mathcal{P}_2(\mathbb{R})$. Define a linear operator $T$ on $\mathcal{P}_2(\mathbb{R})$ by

$$T(p(x)) = xp'(x) - p(1).$$

(a) Find the matrix $\mathcal{M}(T, \beta)$;

(b) Find all the eigenvalues of $T$;

(c) Determine if $T$ is diagonalizable. If so, find an eigenbasis $\alpha$ of $T$ and the corresponding matrix $\mathcal{M}(T, \alpha)$.

a. $T(1) = x(1)' - 1 = -1 = -1 + 0\cdot x + 0\cdot x^2$

$T(x) = x x' - 1 = x - 1 = -1 + 1\cdot x + 0x^2$

$T(x^2) = x(x^2)' - 1^2 = x(2x) - 1 = -1 + 0x + 2x^2$

$\therefore M(T,\beta) = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

b. $M(T,\beta)$ is upper triangular

$\Rightarrow$ diagonal entries are eigenvalues of $T$

$\Rightarrow$ $T$ has eigenvalue $-1, 1, 2$

c. $\dim P_2(\mathbb{R}) = 3$, $T$ has 3 distinct eigenvalues

$\Rightarrow$ $T$ is diagonalizable

Let $A = M(T,\beta)$

For $\lambda = -1$, $A - \lambda I = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$N(A + I) = \text{Span}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow E(-1, T) = \text{span}\{1\}$

For $\lambda = 1$, $A - \lambda I = \begin{bmatrix} -2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$N(A - I) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \Rightarrow E(1, T) = \text{span}\{1 - 2x\}$

For $\lambda = 2$, $A - \lambda I = \begin{bmatrix} -3 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$N(A - 2I) = \text{span} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \Rightarrow E(2, T) = \text{span}\{1 - 3x^2\}$

$\therefore \alpha = \{1, 1-2x, 1-3x^2\}$ is an eigenbasis of $T$

$M(T, \alpha) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

4. (6 pts) Let $V$ be a vector space and $T \in \mathcal{L}(V)$. Suppose $v_1, v_2 \in V$ is an eigenvector of $T$ corresponding to eigenvalue $\lambda_1, \lambda_2$ respectively and $v_1 \neq v_2$. Prove that $v_1 - v_2$ is an eigenvector of $T$ if and only if $\lambda_1 = \lambda_2$.

($\Rightarrow$) If $v_1 - v_2$ is an eigenvector of $T$

Then $\exists \lambda$ s.t. $\lambda(v_1 - v_2) = T(v_1 - v_2)$

$$= T(v_1) - T(v_2)$$

$$= \lambda_1 v_1 - \lambda_2 v_2$$

$\Rightarrow (\lambda - \lambda_1)v_1 + (\lambda_2 - \lambda)v_2 = \vec{0}$

If $\lambda_1 \neq \lambda_2$, then $v_1, v_2$ are lin indept

$$\Rightarrow \lambda - \lambda_1 = \lambda_2 - \lambda = 0$$

$$\Rightarrow \lambda_1 = \lambda = \lambda_2, \text{ contradiction}$$

$\therefore \lambda_1 = \lambda_2$

($\Leftarrow$) If $\lambda_1 = \lambda_2$, then

$$T(v_1 - v_2) = T(v_1) - T(v_2)$$

$$= \lambda_1 v_1 - \lambda_2 v_2$$

$$= \lambda_1 v_1 - \lambda_1 v_2$$

$$= \lambda_1 (v_1 - v_2)$$

$v_1 \neq v_2 \Rightarrow v_1 - v_2 \neq \vec{0} \Rightarrow v_1 - v_2$ is an eigenvector.

5. (9 pts) Let $V = \mathcal{P}_2(\mathbb{R})$ be the vector space of all real polynomials of degree at most 2. Define an inner product on $V$ by

$$\langle p, q\rangle = p(0)q(0) + 2\,p(1)q(1) + p(2)q(2) \quad \text{for any } p, q \in V. \qquad (*)$$

(a) Apply the Gram-Schmidt Process to $\{1, x\} \subset V$ to obtain an orthonormal list.

(b) Does the formula $(*)$ define an inner product on $\mathcal{P}_3(\mathbb{R})$, the vector space of all real polynomials of degree at most 3? Justify your answer.

a. let $v_1 = 1$, $v_2 = x$

let $u_1 = v_1$, $\|u_1\| = \sqrt{\langle u_1, u_1\rangle}$

$= \sqrt{\langle 1, 1\rangle}$

$= \sqrt{(1)(1) + 2(1)(1) + (1)(1)}$

$= 2$

let $e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{2}$

let $u_2 = v_2 - \langle v_2, e_1\rangle e_1$

$= x - \langle x, \frac{1}{2}\rangle \cdot \frac{1}{2}$

$= x - \left[(0)(\frac{1}{2}) + 2(1)(\frac{1}{2}) + (2)(\frac{1}{2})\right]\frac{1}{2}$

$= x - 1$

$\|u_2\| = \sqrt{\langle x-1, x-1\rangle} = \sqrt{(-1)(-1) + 2(0)(0) + (1)(1)} = \sqrt{2}$

let $e_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{2}}(x-1)$

$\therefore$ $\frac{1}{2}$, $\frac{1}{\sqrt{2}}(x-1)$ is an orthonormal basis

b. No.

Let $p(x) = x(x-1)(x-2) \in P_3(\mathbb{R})$

Then $p(x) \neq \vec{0}$ but

$$\langle p, p \rangle = p(0)p(0) + 2p(1)p(1) + p(2)p(2)$$

$$= (0)(0) + 2(0)(0) + (0)(0)$$

$$= 0$$

$\therefore \langle \, , \rangle$ is not positive definite on $P_3(\mathbb{R})$

—END OF TEST 2—