Math4230 Exercise 4

1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a bounded convex function. Show that f is constant.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Show that if $x_1 < x_2 < x_3$, then
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

3. (a) A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called *quasiconvex* if
$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}, \ \forall x, y, \ \lambda \in [0,1]$$
Show that a function is quasiconvex if and only if the level set $\{x|f(x) \le a\}$ is convex for all $a \in \mathbb{R}$.
   (b) Show that every convex function is quasiconvex. Is the converse true?

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Show that the following are equivalent.
   (a) All level sets of f are compact, i.e. $\{x| \ f(x) \le a\}$ is compact for all a.
   (b) f is coercive, i.e. for all sequence $\{x_k\}$ with $||x_k|| \to \infty, \ f(x_k) \to \infty$