Math4230 Exercise 11

1. Consider the convex problem

$$\min f(x) \text{ subject to } g_i(x) \le 0,\ i = 1, ..., m$$

   Assume that $x^* \in \mathbb{R}^n,\ \lambda^* \in \mathbb{R}^m$ satisfy the KKT conditions

$$g_i(x^*) \le 0,\ i = 1, ...m$$

$$\lambda_i^* \ge 0,\ i = 1, ..., m$$

$$\lambda_i^* g_i(x^*) = 0,\ i = 1, ..., m$$

$$\nabla f(x^*) + \sum \lambda_i^* \nabla g_i(x^*) = 0$$

   Show that

$$\langle \nabla f(x^*), x - x^* \rangle \ge 0$$

   for all feasible x.

2. Consider

$$\max_{(x,y,z)\in\mathbb{R}^3} 2x + 3y + 2z$$

$$\text{subject to } x^2 + y^2 + z^2 = 1,\ x + y + z \ge 0$$

   (a) Show that an optimal solution exists.
   (b) Write down the KKT conditions.
   (c) Solve the KKT conditions. (I forgot that the equality constraint is not affine. We need constraint qualifications to solve this problem.)

3. Let A be an $n \times n$ real symmetric matrix. Consider

$$\min_{||x||=1} \langle x, Ax \rangle$$

   (a) Write down the KKT conditions.
   (b) Assuming the KKT conditions are necessary and sufficient, show that the optimal value is an eigenvalue of A.

4. Consider

$$\min_{x\in\mathbb{R}} x$$

$$\text{subject to } x^2 \le 0$$

   (a) Write down the dual problem. Hence, show that there is no duality gap.
   (b) Show that there is no dual optimal solution.