Feb. 24:

Chapter 2: Stationary Distributions

SD

* What's SD?
* Does it exist?
  - If exists, unique? how to find it?
  - If not, what's the reason?

§1. SD

What's SD?

MC: $\{X_n\}_{n\geq 1}$, S, P

S: state space

P: transition matrix

expect: find a distribution π (prob. row vector)

s.t. if the pdf of X_0 is π, then the pdf of X_n for all $n \geq 1$ is the same as π.

X_0	X_1	X_2	
0	1	2	
π	π	π	......
	$\parallel$		
	πP		

$\pi(k) = P(X_0 = k), \forall k \in S$

Recall: $\pi^{(n)}$ = prob. row vector to denote pdf of X_n, then

$$\pi^{(n)} = \pi^{(n-1)} P$$

Def. π is a SD if

(i) π is a prob. (row-) vector: i.e.

$$0 \le \pi(x) \le 1, \quad \forall x \in S$$

$$\sum_{x \in S} \pi(x) = 1.$$

(ii) (Stationary) π satisfies $\pi = \pi P$: "vector form"

$$\pi(y) = \sum_{x \in S} \pi(x) P(x, y), \quad \forall y \in S.$$

How to find SD π? By def.: π is a left 1-eVector

i.e. $\pi P = \pi$

just a linear system of equations for unknown $\pi(x)$, $x \in S$.

Solve it!

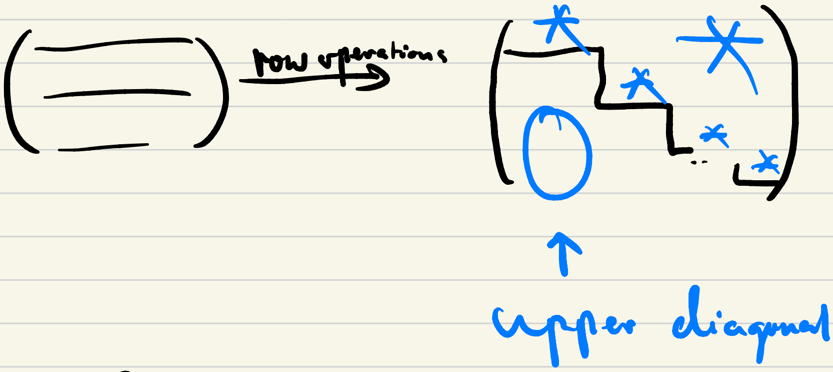
$$P^T \pi^T = \pi^T \Rightarrow \underbrace{(P^T - I)}\pi^T = 0$$

homogeneous system

look for solutions π

as prob. row vectors

Coefficient matrix

$P^T - I =$ 

A special situation (S is finite)

Claim: If the limit distribution exists, i.e.

$$\lim_{n\to\infty} P(X_n = x) = \underbrace{\pi(x)}_{\in[0,1]}, \quad \forall x \in S$$

$$\boxed{\sum_{x\in S} P(X_n = x) = 1 \;\xrightarrow{n\to\infty}\; \sum_{x\in S} \pi(x) = 1}$$

or in vector form

$$\lim_{n\to\infty} \pi^{(0)} P^n = \pi$$

then

$$\pi = \left[\lim_{n\to\infty} \pi^{(0)} P^{n-1}\right] \cdot P$$

$$= \pi P$$

$\therefore$ π is a SD.

Another view:

If $\lim_{n\to\infty} P^n(x,y) \left[= \lim_{n\to\infty} P_x(X_n = y)\right] = \pi(y)$, $\forall x, y \in S$

row prob. vector

or in vector form

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$$

then π is a SD

(pf. $\lim_{n\to\infty} \pi^{(n)} = \lim_{n\to\infty} \pi^{(0)} P^n$

$$= \pi^{(0)} \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$$

$$= \pi \;)$$

Reminder: (when S is finite)

If $P = QDQ^{-1}$ (diagonanizable

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}$$

$$Q = \begin{bmatrix} \Big| & \Big| & \cdots & \Big| \end{bmatrix}$$

e-vectors associated with each λ_i

then $P^n = Q \begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_N^n \end{bmatrix} Q^{-1}$

$$\lim_{n\to\infty} P^n = Q \begin{bmatrix} \lim_{n\to\infty} \lambda_1^n & & \\ & \ddots & \\ & & \lim_{n\to\infty} \lambda_N^n \end{bmatrix} Q^{-1}$$

#

Prop. (Sufficient conditions + finite state space case)

Let $\{X_n\}_{n=0}^{\infty}$ be a MC with Markov matrix P over the finite state space S. For P, assume:

must exist: $P\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}=1\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$

(i) the left 1-eigenvector can be chosen to have all ≥ 0 entries;

(ii) 1 is a simple eigenvalue;

(iii) all other eigenvalues $|\lambda_i| < 1$.

Then, the chain has a unique SD π, and

$$\lim_{n\to\infty} P^n = \begin{pmatrix}\pi\\ \pi\\ \vdots\\ \pi\end{pmatrix}. \quad \#$$

Proof. (sketch)

$$P = QDQ^{-1}, \qquad D = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & M \end{array}\right]$$

$$Q = \left[\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix} \Big| \ [\ ] \ \cdots \ [\ ]\right]$$

$$Q^{-1} = \begin{bmatrix} \pi \\ \text{---} \\ \vdots \\ \text{---} \end{bmatrix}$$

1st row is a prob. vector given by π

$$\lim_{n\to\infty} M^n = 0$$

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty} \left[Q D Q^{-1} \right]^n$$

$$= \lim_{n\to\infty} Q D^n Q^{-1}$$

$$= Q \left(\lim_{n\to\infty} D^n \right) Q^{-1}$$

$$= Q \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right] \boxed{Q^{-1}}}_{= \begin{bmatrix} \pi \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \qquad Q^{-1} = \begin{bmatrix} \pi \\ \text{---} \\ \vdots \\ \text{---} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix} & \cdots \end{bmatrix} \begin{bmatrix} \pi \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}. \qquad \#$$

Rk: For detailed proof, refer to Lawler's textbook.

Note:

(1) Without (ii), "uniqueness" may fail:

e.g., $P = \left[\begin{array}{c|c} P_1 & O \\ \hline O & P_2 \end{array}\right]$

$\pi_1 P_1 = \pi_1 \Leftarrow \pi_1$: SD of P_1

$\pi_2 P_2 = \pi_2 \Leftarrow \pi_2$: SD of P_2

$\Rightarrow$ $0 \le \lambda \le 1$, $[\lambda \pi_1, (1-\lambda)\pi_2]$ is a SD of P

{ check:

$[\lambda\pi_1, (1-\lambda)\pi_2] \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$

$= [\lambda \pi_1 P_1, (1-\lambda)\pi_2 P_2]$

$= [\lambda\pi_1, (1-\lambda)\pi_2]$ }

And, 1 is NOT a simple e-value of P,

$\det(P-\lambda I) = \underline{\det(P_1 - \lambda I)}\ \underline{\det(P_2 - \lambda I)}$

$= (\lambda-1)\,\boxed{} \times (\lambda-1)\,\boxed{}$

$= (\lambda-1)^2\,\boxed{}\,.\ )$

(2) Without (iii), existence of

$\lim_{n\to\infty} P^n$ may not exist.

$\exists$ an example s.t. $\begin{cases} \exists \text{ a SD} \\ \text{but } \cancel{\lim_{n\to\infty} P^n} \end{cases}$

e.g.

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{e-values} = \pm 1$$

$$\therefore\ |-1| = 1 \not< 1$$

$\lim_{n\to\infty} P^n$ does NOT exist

$$P^n = \begin{cases} P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } n \text{ odd} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } n \text{ even} \end{cases}$$

Howerver

$$\left[\tfrac{1}{2}, \tfrac{1}{2}\right] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left[\tfrac{1}{2}, \tfrac{1}{2}\right]$$

$\therefore \; [\frac{1}{2}, \frac{1}{2}]$ is a SD of P.

Two important facts for finite state space S

(without proof ; proof non-trivial ; refer to the Lawler's textbook)

Fact 1 If P^n for some $n \geq 1$ has all entries strictly positive then those three conditions in the previous prop. are satisfied,

$$\therefore \; \exists ! \text{ SD } \pi \;\; \& \;\; \lim_{n\to\infty} P^n = \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{pmatrix} \quad . \#$$

Fact 2 If P is irreducible then P has a unique SD, (but $\lim_{n\to\infty} P^n$ may not exist)

Compute SD of P is no longer irreducible, i.e. P can be reducible, but still the state space is finite.

Dicuss :

$\lim_{n\to\infty} P^n$: long-run behavior

First, see a simple situation :

$$\underset{\substack{\uparrow \\ \text{State space} \\ \text{(finite)}}}{S} = \underset{\substack{\uparrow \\ \text{irreducible closed set} \\ \text{of recurrent states}}}{C_1} \cup C_2 \cup S_T$$

P <u>Canonical form</u>

$$\begin{array}{c|c} & \begin{array}{ccc} \boxed{C_1} & \boxed{C_2} & \boxed{S_T} \end{array} \\ \begin{array}{c} \boxed{C_1} \\ \boxed{C_2} \\ \boxed{S_T} \end{array} & \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ S_1 & S_2 & Q \end{bmatrix} \end{array}$$

$$P^n = \begin{array}{c|c} & \begin{array}{ccc} \boxed{C_1} & \boxed{C_2} & \boxed{S_T} \end{array} \\ \begin{array}{c} \boxed{C_1} \\ \boxed{C_2} \\ \boxed{S_T} \end{array} & \begin{bmatrix} P_1^n & 0 & 0 \\ 0 & P_2^n & 0 \\ \underset{\sim}{S_{1n}} & \underset{\sim}{S_{2n}} & Q^n \end{bmatrix} \end{array} \xrightarrow[n\to\infty]{} \ ?$$

<u>Assume</u>:

<u>P_1 has a unique SD π_1</u> $+$ $\lim_{n\to\infty} P_1^n = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_1 \end{pmatrix}$

<u>P_2 has a unique SD π_2</u> $+$ $\lim_{n\to\infty} P_2^n = \begin{pmatrix} \pi_2 \\ \vdots \\ \pi_2 \end{pmatrix}$

First notice:

$$\lim_{n\to\infty} Q^n = 0$$

pf.

$$\lim_{n\to\infty} Q^n(x,y) \overset{\substack{x\in S_T \\ y\in S_T}}{=\!=} \lim_{n\to\infty} P^n(x,y)$$

$(x, y \in S_T$
S: finite-states $)$

$$= 0. \quad \#$$

Assume:

$$\lim_{n\to\infty} S_{1n} = \underset{\sim}{A_1}\ ?$$

$$\lim_{n\to\infty} S_{2n} = \underset{\sim}{A_2}\ ?$$

$$A_1(x,y) \;\underset{=\!=}{\overset{x\in S_T,\ y\in C_1}{=\!=}}\; \lim_{n\to\infty} P^n(x,y)$$

prob that the chain from a transient $x \in S_T$ visits a recurrent state $y \in C_1$ in the long run

$$A_2(x,y) \;\underset{=\!=}{\overset{x\in S_T,\ y\in C_2}{=\!=}}\; \underbrace{\lim_{n\to\infty} P^n(\underset{\in S_T}{x}, \underset{\in C_2}{y})}_{\text{long-run prob.}}$$

Claim: If $S_T = \{x_1, \dots, x_\ell\}$

then

$$A_1 \left(= \lim_{n\to\infty} P^n(\underset{S_T}{\underset{\downarrow}{x}}, \underset{C_1}{\underset{\downarrow}{y}})\right) = \begin{array}{c|c} & C_1 \\ \hline x_1 & \boxed{\rho_{C_1}(x_1)\,\pi_1} \\ x_2 & \boxed{\rho_{C_1}(x_2)\,\pi_1} \\ \vdots & \vdots \\ x_\ell & \boxed{\rho_{C_1}(x_\ell)\,\pi_1} \end{array} \quad \underset{\sim}{\#S_T \times \#C_1} = \ell \times \#C_1$$

$$A_2 \left(= \lim_{n\to\infty} P^n(\underset{S_T}{\underset{\downarrow}{x}}, \underset{C_2}{\underset{\downarrow}{y}})\right) = \begin{array}{c|c} & C_2 \\ \hline x_1 & \boxed{\rho_{C_2}(x_1)\,\pi_2} \\ x_2 & \boxed{\rho_{C_2}(x_2)\,\pi_2} \\ \vdots & \vdots \\ x_\ell & \boxed{\rho_{C_2}(x_\ell)\,\pi_2} \end{array} \quad \ell \times \#C_2$$

Proof. omitted.

A very special case: all recurrent states are absorbing

$$\underset{\substack{\uparrow \\ \text{Canonical} \\ \text{form}}}{P} = \begin{array}{c|cc} & \boxed{S_R} & \boxed{S_T} \\ \hline \boxed{S_R} & I_{\#S_R \times \#S_R} & O \\ \boxed{S_T} & S & Q \end{array}$$

$$P^n = \begin{array}{c|cc} & \overset{j \in}{\boxed{S_R}} & \\ \boxed{S_R} & I & 0 \\ i \in \boxed{S_T} & S_n & Q^n \end{array} \xrightarrow[n\to\infty]{} \left[\begin{array}{c|c} I & 0 \\ \hline A & 0 \end{array}\right]$$

$$A \overset{\text{''}}{=} (I-Q)^{-1}S$$

$\lim_{n\to\infty} Q^n = 0$. $\quad \lim_{n\to\infty} S_n \overset{\text{assume}}{=} A$

how to compute it.

The meaning of $A(i,j)$

$$\overset{i\in S_T}{\underset{j \in \underbrace{S_R}_{j \text{ is absorbing}}}{=\!=}} \lim_{n\to\infty} P^n(\overset{\text{transient}}{i}, \overset{\text{absorbing}}{j})$$

$$= P_i(T_j < \infty) = \rho_{ij}$$

Recall:

$$\underbrace{\rho_{ij}}_{A_{ij}} \overset{i \in S_T}{\underset{j \in S_R}{=\!=}} \underbrace{P(i,j)}_{S_{ij}} + \sum_{k \in S_T} \underbrace{P(i,k)}_{Q_{ik}} \underbrace{\rho_{kj}}_{A_{kj}}$$

Component form $\Downarrow$ matrix form

$$\underline{A} = \underline{\underline{S}} + \underline{\underline{Q}}\,\underline{A}$$

? given given ?

$$\therefore \quad IA - QA = S$$

$$\underbrace{(I-Q)}_{\text{invertible}}A = S$$

$$\therefore \quad \boxed{A = (I-Q)^{-1}S}$$

general situation

$$\lim_{n\to\infty} P^n = ?$$

P original Markov matrix $\xrightarrow[S = S_T \cup (\bigcup_{i=1}^{k} C_i)]{\text{find}}$ Canonical form

S finite

$$\bar{P} = \begin{array}{c|c} & \begin{array}{cccc} C_1 & \cdots & C_k & S_T \end{array} \\ \begin{array}{c} C_1 \\ \vdots \\ C_k \\ S_T \end{array} & \left[\begin{array}{ccc|c} P_1 & & & \\ & \ddots & & \\ & & P_k & \\ \hline \boxed{S_1} & \cdots & \boxed{S_k} & Q \end{array}\right] \end{array}$$

S_T: $x_1, \dots, x_\ell$

$$\lim_{n\to\infty} \bar{P}^n = \begin{array}{c|c} & \begin{array}{cccc} C_1 & \cdots & C_k & S_T \end{array} \\ \begin{array}{c} C_1 \\ \vdots \\ C_k \\ S_T \end{array} & \left[\begin{array}{c|c|c|c} \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_1 \end{bmatrix} & 0 & 0 & 0 \\ \hline 0 & \ddots & 0 & 0 \\ \hline 0 & 0 & \begin{bmatrix} \pi_k \\ \vdots \\ \pi_k \end{bmatrix} & 0 \\ \hline A_1 & \cdots & A_k & 0 \end{array}\right] \end{array}$$

$$\{x_1, \dots, x_\ell\} \overset{\text{def.}}{=} S_T$$

$$A_1 = \begin{bmatrix} \rho_{C_1}(x_1)\pi_1 \\ \vdots \\ \rho_{C_1}(x_\ell)\pi_1 \end{bmatrix}_{\ell \times \# C_1}, \quad \dots, \quad A_k = \begin{bmatrix} \rho_{C_k}(x_1)\pi_k \\ \vdots \\ \rho_{C_k}(x_\ell)\pi_k \end{bmatrix}_{\ell \times \# C_k}$$

Point: Find

$S_T = \{x_1, \dots, x_\ell\}$

$$\hat{A} = \begin{array}{c} x_1 \\ \vdots \\ x_\ell \end{array} \begin{bmatrix} \rho_{C_1}(x_1) & \rho_{C_2}(x_1) & \cdots & \rho_{C_k}(x_1) \\ \vdots & \vdots & & \vdots \\ \rho_{C_1}(x_\ell) & \rho_{C_2}(x_\ell) & \cdots & \rho_{C_k}(x_\ell) \end{bmatrix}_{\ell \times k}$$

Recall:

$$\underbrace{\rho_{C_j}(x_i)}_{\hat{A}_{ij}} \overset{\substack{1 \le i \le \ell \\ x_i \in S_T}}{\underset{\substack{C_j \\ 1 \le j \le k}}{=\!=}} \underbrace{\sum_{y \in C_j} P(x_i, y)}_{\text{For instance}} + \underbrace{\sum_{x_m \in S_T}}_{= \sum_{m=1}^{\ell}} \underbrace{P(x_i, x_m)}_{Q_{im}} \underbrace{\rho_{C_j}(x_m)}_{\hat{A}_{mj}}$$

For instance fix $i \in \{1, \dots, \ell\}$

$= \boxed{/////}$ Sum of i^{th} row of S_j

$$\underbrace{\sum_{m=1}^{\ell} Q_{im} \hat{A}_{mj}} = (Q\hat{A})_{ij}$$

matrix form $\Rightarrow$

$$\hat{A} = (I - Q_{\ell \times \ell})^{-1} \begin{array}{cccc} 1^{st} & & & k^{th} \\ \end{array} \begin{bmatrix} \boxed{////} & & & \\ \square & & & \\ \vdots & & \cdots & \\ \square & & & \end{bmatrix}_{\ell \times k}$$

Sum of 1^{st} row of S_1

example: Refer to tutorials in the next Mon.